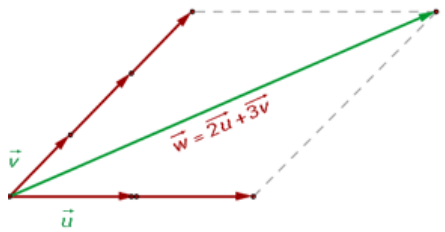
Friday, September 14, 2018

6.8 Linear Combinations and Spanning Sets

Linear Combination of Vectors:

For non-collinear vectors, $\vec{u}$ and $\vec{v}$, a linear combination of these vectors is $a\vec{u} + b\vec{v}$, where a and b are scalars. The vector $a\vec{u} + b\vec{v}$ is the diagonal of the parallelogram formed by $a\vec{u}$ and $b\vec{v}$.



We have seen that any vector in R^2 can be written as a linear combination of $\vec{i}$ and $\vec{j}$ ($\vec{OP} = (a, b) = a\vec{i} + b\vec{j}$), and that any vector in R^3 can also be written as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Spanning Sets in R^2

Because every vector in R^2 can be written uniquely as a linear combination of i and j, {i, j} is considered a spanning set for R^2. A "spanning set" refers to a vector space that contains all vectors in the plane containing the two vectors you are combining (the xy plane in the case of R^2).

In general, any two non - zero, non-collinear vectors $\vec{u}$ and $\vec{v}$ will span R^2, and any vector, $\vec{w}$, in this plane can be written as a linear combination of $\vec{u}$ and $\vec{v}$. That means that $\vec{w} = a\vec{u} + b\vec{v}$ for scalars a and b.

Example: Choose any two non - zero, non - collinear vectors in R^2. Then write (-3, 5) as a linear combination of your two vectors.

$\vec{u} = (3, 2)$ $\quad \vec{w} = (-3, 5)$

$\vec{v} = (4, 1)$ $\quad a(3, 2) + b(4, 1) = (-3, 5)$

$3a + 4b = -3$ ① $\quad 2a + b = 5$ ②

$b = \boxed{5 - 2a}$

$3a + 4(5 - 2a) = -3$

$3a + 20 - 8a = -3$

$\frac{-5a}{-5} = \frac{-23}{-5}$

$a = \boxed{\frac{23}{5}}$

$b = 5 - 2\left(\frac{23}{5}\right)$

$= \frac{25}{5} - \frac{46}{5}$

$b = -\frac{21}{5}$

Linear Comb

$\frac{23}{5}(3, 2) - \frac{21}{5}(4, 1) = (-3, 5)$

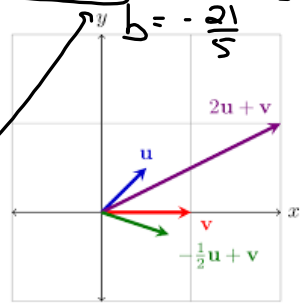


Figure 1: Vector combinations.

Spanning Sets in R^3

Vectors are coplanar if, when they are arranged tail - to - tail, their heads lie in the same plane. Three vectors are coplanar if one can be expressed as a linear combination of the other two ($\vec{w} = a\vec{u} + b\vec{v}$).

Any pair of non-zero, non-collinear vectors will span a plane in R^3.

Example: Determine if the vectors $\vec{a} = (1, 2, 3)$, $\vec{b} = (2, -1, 3)$, and $\vec{c} = (8, 1, 5)$ are coplanar.

$a(1,2,3) + b(2,-1,3) = (8,1,5)$??

① $a + 2b = 8$
$a = \boxed{8-2b}$
$a = 8 - 2(3)$
$a = 2$

② $2a - b = 1$
$2(8-2b) - b = 1$
$16 - 4b - b = 1$
$-5b = -15$
$b = 3$

③ $3a + 3b = 5$
Check $a = 2$, $b = 3$ in ②
$3(2) + 3(3) \stackrel{?}{=} 5$
$6 + 9 = 5$
$15 \neq 5$
$\therefore$ These 3 vectors are not coplanar

3 vectors are coplanar if the values of a+b for 2 equations satisfy the 3rd.

Example: Determine the value for x such that the points A(-1, 3, 4), B(-2, 3, -1) and C (-5, 6, x) all lie on a plane that contains the origin. (0,0,0)

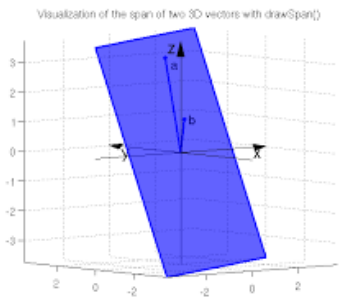
coplanar

$\vec{AO} = (1, -3, -4)$
$\vec{BO} = (2, -3, 1)$
$\vec{CO} = (5, -6, -x)$

$a(1,-3,-4) + b(2,-3,1) = (5,-6,-x)$

① $a + 2b = 5$
$a = \boxed{5-2b}$
$a = 5 - 2(3)$
$a = -1$

② $-3a - 3b = -6$
$-3(5-2b) - 3b = -6$
$-15 + 6b - 3b = -6$
$3b = 9$
$b = 3$

③ $-4a + b = -x$
$-4(-1) + 3 = -x$
$\boxed{-7 = x}$

$\therefore C(-5, 6, -7)$