Calculus Appendix Part 1: Implicit Differentiation

Until now, you have worked primarily with functions that are written so that y is explicitly defined in terms of x. Some examples are given below.

$$y = \sqrt{2x-5} \qquad f(x) = \frac{8x}{x^3-5} \qquad y = 2sinx$$

In the event that you saw a function defined in terms of two variables, until now you would just rearrange it to write it as a function of y in terms of x.

a) Rewrite $x^2 + y^2 = 25$ in terms of y.

We can differentiate this function, as long as we identify the correct portion to use when we are asked to determine the slope of a tangent at a given point.

b) Determine the slope of the tangent to the curve in a) when x = 3. What issue have you run in to? How many answers do you need to find?

Rather than rearranging to create an explicit relationship, we can use **implicit differentiation** to differentiate an equation with respect to x. To do this, we differentiate both sides of the equation with respect to x. We need to remember our derivative rules (product, quotient, chain, etc.) and Leibniz notation for the chain rule ($\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$) so that we do this properly.

Implicit Differentiation: Find the derivative of $x^2 + y^2 = 25$.

Notes:

- Take the derivative of both sides with respect to x.
- Apply the chain rule when working with terms with 'y'.
- Isolate dy/dx to get the derivative.

The example that we did can be completed implicitly or explicitly. However, there are lots of functions that cannot be expressed explicitly.

For example, find the derivative of $2xy - y^3 = 4$.

Because these derivatives are dependent on both x and y, we need to sub in values for both if we are finding the tangent slope (use the whole point of tangency).

Practice Problems:

PART A

1. State the chain rule. Outline a procedure for implicit differentiation.
2. Determine $\frac{dy}{dx}$ for each of the following in terms of x and y, using implicit differentiatio
   a. $x^2 + y^2 = 36$
   b. $15y^2 = 2x^3$
   c. $3xy^2 + y^3 = 8$
   d. $9x^2 - 16y^2 = -144$
   e. $\frac{x^2}{16} + \frac{3y^2}{13} = 1$
   f. $x^2 + y^2 + 5y = 10$
3. For each relation, determine the equation of the tangent at the given point.
   a. $x^2 + y^2 = 13, (2, -3)$
   b. $x^2 + 4y^2 = 100, (-8, 3)$
   c. $\frac{x^2}{25} - \frac{y^2}{36} = -1, (5\sqrt{3}, -12)$
   d. $\frac{x^2}{81} - \frac{5y^2}{162} = 1, (-11, -4)$

We will work on these here this week and next week, and then I'll move on to Related Rate problems :)

PART B

4. At what point is the tangent to the curve $x + y^2 = 1$ parallel to the line $x + 2y = 0$?
5. The equation $5x^2 - 6xy + 5y^2 = 16$ represents an ellipse.
   a. Determine $\frac{dy}{dx}$ at $(1, -1)$.
   b. Determine two points on the ellipse at which the tangent is horizontal.
6. Determine the slope of the tangent to the ellipse $5x^2 + y^2 = 21$ at the point $A(-2, -1)$.
7. Determine the equation of the normal to the curve $x^3 + y^3 - 3xy = 17$ at the point $(2, 3)$.
8. Determine the equation of the normal to $y^2 = \frac{x^3}{2 - x}$ at the point $(1, -1)$.
9. Determine $\frac{dy}{dx}$.
   a. $(x + y)^3 = 12x$
   b. $\sqrt{x + y} - 2x = 1$
10. The equation $4x^2y - 3y = x^3$ implicitly defines y as a function of x.
    a. Use implicit differentiation to determine $\frac{dy}{dx}$.
    b. Write y as an explicit function of x, and compute $\frac{dy}{dx}$ directly.
    c. Show that your results for parts a. and b. are equivalent.