Date: ____________________

Vector Appendix: Gaussian Elimination

In Chapter 9, we will need to be able to come up with solutions for systems of equations of planes. Recall that the Cartesian equation of a plane is $Ax + By + Cz + D = 0$, where (A, B, C) is the normal vector to the plane and D is a constant, and (x, y, z) is any point on the plane.

A Quick Review: Solving a Linear System of Equation with Elimination

Solve the linear system of equations represented by $2x - 5y = 4$ and $3x + 2y = -1$ using elimination. What are we solving for?

↳ POI

① $2x - 5y = 4$ ② $3x + 2y = -1$

①×3: $6x - 15y = 12$

②×2: $6x + 4y = -2$ (subtract)

$-19y = 14$

$y = \frac{-14}{19}$

Sub $y = \frac{-14}{19}$ into ②

$3x + 2\left(\frac{-14}{19}\right) = -1$

$x = \frac{9}{57}$

POI $\left(\frac{9}{57}, \frac{-14}{19}\right)$

How many unknowns do we have in the Cartesian equation of a plane? As a result, how many equations would we need to be able to solve the system of equations? 3 variables, so we need 3 equations

This can be done using standard algebraic techniques (I'll demonstrate that in section 9.2), but it is easier for us to use matrices and elementary row operations (Gaussian elimination).

System of Equations:

$$x + 2y + 2z = 9$$
$$x + y = 1$$
$$2x + 3y - z = 1$$

Coefficient Matrix:

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \end{array}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right]$$

How are these representations related to one another? What do you think a matrix is? How is an augmented matrix different from a coefficient matrix?

↳ an organized table of coefficients

contains the right side of the equations

The benefit of using matrices is that they provide us with a systematic, organized method for solving systems of equations. This is especially useful for more larger systems, but can also be used with linear equations.

To solve with matrices, we apply operations to the rows of the matrix (elementary row operations) until we manage to reduce one of the rows to an equation we can solve.

<u>Elementary Row Operations</u>

1. Multiply a row by a nonzero constant.
2. Interchange any pair of rows.
3. Add a multiple of one row to (or subtract a multiple of one row from) another row to replace the second row.

*When you carry out row operations, you only change one of the two rows that you are working, and the end goal is to have a row in the augmented matrix that only has two numbers left in it.

An example is the easiest way to see this in action, so let's solve!

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right] \xrightarrow{R_3 - 2R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

want zero.

we want zero here

want 0 here

$R_2 + R_3 \to R_3$

$$\begin{array}{ccc|c} x & y & z & \\ \hline \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

Solve this!

$-3z = -9$

$z = 3$

$-y - 2z = -8$

$-y - 2(3) = -8$

$-y = -2$

$y = 2$

$x + 2y + 2z = 9$

$x + 2(2) + 2(3) = 9$

$x + 10 = 9$

$x = -1$

POI $(-1, 2, 3)$

<u>Tips to stay organized (and to help me follow your work)</u>:

1. Clearly label the row operations that you are applying, and indicate the row that you are writing the result in (ex/ $2R_1 + R_2 \to R_2$).
2. Be extremely careful when you are inputting your coefficients so that you don't end up working with unlike terms.
3. Show your work once you reach the point where you can solve using the equations. This will help you to catch errors!
4. It is okay to combine steps when you are working in your matrix, just be sure to include all of the operations that you used in your list.

The end goal of Gaussian elimination is to produce a matrix in "row-echelon form". This just means that you have any row that consists of all zeros at the bottom of the matrix, and all non-zero rows have their first non-zero number further to the right than the row before them.

Row Echelon Form: $\left[\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$

Example: Solve the system of equations provided using elementary row operations.

R_1 $2x + y + 2z = 2$

R_2 $x + y + z = 2$

R_3 $x + 2y - 3z = 4$

$$\begin{array}{cc} & \begin{array}{cccc} x & y & z & c \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \left[\begin{array}{ccc|c} 2 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & -3 & 4 \end{array}\right] \end{array} \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & -4 & 2 \end{array}\right] \xrightarrow{2R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -4 & 2 \end{array}\right]$$

solve!

$y = 2$

From R_3

$y - 4z = 2$

$2 - 4z = 2$

$-4z = 0$

$z = 0$

From R_1

$2x + y + 2z = 2$

$2x + 2 = 2$

$2x = 0$

$x = 0$

$(0, 2, 0)$

Note: When you are doing today's homework, ignore questions about HOW the planes intersect. You are just trying to master the skill of solving using Gaussian elimination!