3.1 Higher Order Derivatives, Velocity + Acceleration

1. $v(1) = 1\text{ m/s}$
$v(5) = -15\text{ m/s}$ } opposite directions.

2a) $y = x^{10} + 3x^6$
$y' = 10x^9 + 18x^5$
$y'' = 90x^8 + 90x^4$

b) $f(x) = \sqrt{x}$
$f'(x) = \frac{1}{2}x^{-1/2}$
$f''(x) = -\frac{1}{4}x^{-3/2}$

c) $y = (1-x)^2$
$y' = 2(1-x)(-1)$
$= -2(1-x)$
$= -2 + 2x$
$y'' = 2$

d) $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$
$h'(x) = 12x^3 - 12x^2 - 6x$
$h''(x) = 36x^2 - 24x - 6$

e) $y = 4x^{3/2} - x^{-2}$
$y' = 6x^{1/2} + 2x^{-3}$
$y'' = 3x^{-1/2} - 6x^{-4}$
$= \frac{3}{\sqrt{x}} - \frac{6}{x^4}$

f) $f(x) = \frac{2x}{x+1}$
$f'(x) = \frac{2(x+1) - 2x(1)}{(x+1)^2}$
$= \frac{2}{(x+1)^2}$
$f''(x) = \frac{-2[2(x+1)(1)]}{(x+1)^4}$
$= \frac{-4}{(x+1)^3}$

2g) $y = x^2 + x^{-2}$
$y' = 2x - 2x^{-3}$
$y'' = 2 + 6x^{-4}$
$= 2 + \frac{6}{x^4}$

h) $g(x) = (3x-6)^{1/2}$
$g'(x) = \frac{1}{2}(3x-6)^{-1/2}(3)$
$= \frac{3}{2}(3x-6)^{-1/2}$
$g''(x) = -\frac{3}{4}(3x-6)^{-3/2}(3)$
$= \frac{-9}{4(3x-6)^{3/2}}$

i) $y = (2x+4)^3$
$y' = 3(2x+4)^2(2)$
$= 6(2x+4)^2$
$y'' = 12(2x+4)(2)$
$= 24(2x+4)$
$= 48x + 96$

j) $h(x) = x^{5/3}$
$h'(x) = \frac{5}{3}x^{2/3}$
$h''(x) = \frac{10}{9}x^{-1/3}$
$= \frac{10}{9x^{1/3}}$

3a) $s(t) = 5t^2 - 3t + 15$
$s'(t) = 10t - 3$ $(v(t))$
$s''(t) = 10$ $(a(t))$

b) $s(t) = 2t^3 + 36t - 10$
$s'(t) = 6t^2 + 36$ $(v(t))$
$s''(t) = 12t$ $(a(t))$

c) $s(t) = t - 8 + 6t^{-1}$
$s'(t) = 1 - 6t^{-2}$
$s''(t) = 12t^{-3}$

d) $s(t) = (t-3)^2$
$s'(t) = 2(t-3)$
$s''(t) = 2$

e) $s(t) = (t+1)^{1/2}$
$s'(t) = \frac{1}{2}(t+1)^{-1/2}$
$s''(t) = -\frac{1}{4}(t+1)^{-3/2}$

f) $s(t) = \frac{9t}{t+3}$
$s'(t) = \frac{9(t+3) - 9t}{(t+3)^2}$
$= \frac{27}{(t+3)^2}$
$s''(t) = \frac{-27(2(t+3))}{(t+3)^4}$
$= \frac{-54}{(t+3)^3}$

4a) i) $t=3$
ii) $1<t<3$
iii) $3<t<5$

b) i) $t=3, t=7$
ii) $1<t<3,\ t>7$
iii) $3<t<7$

5a) $s(t)=\frac{1}{3}t^3-2t^2+3t$
$s'(t)=t^2-4t+3$
$s''(t)=2t-4$

b) $t^2-4t+3=0$
$(t-3)(t-1)=0$
$t=3,\ t=1$

c) $\frac{1}{3}t(t^2-6t+9)=0$
$\frac{1}{3}t(t-3)^2=0$
$t=0,\ t=3$
Returns at $t=3$

6a) $s(t)=-\frac{1}{3}t^2+t+4$
$s'(t)=-\frac{2}{3}t+1$
$s'(1)=\frac{1}{3}$ (+)
$s'(4)=-\frac{5}{3}$ (−)
∴ (+) direction at $t=1$,
(−) direction at $t=4$

b) $s(t)=t(t-3)^2$
$s'(t)=2t(t-3)+(t-3)^2$
$=(t-3)(2t+t-3)$
$=(t-3)(3t-3)$
$=3(t-3)(t-1)$
$s'(1)=0$ (at rest)
$s'(4)=9$ (+) direction.

6c) $s(t)=t^3-7t^2+10t$
$s'(t)=3t^2-14t+10$
$s'(1)=-1$ (−) direction
$s'(4)=2$ (+) direction

7. $s(t)=t^2-6t+8$
a) $s'(t)=2t-6$
b) $2t-6=0$
$t=3$

8. $s(t)=40t-5t^2$
a) $s'(t)=40-10t$
$40-10t=0$
$\boxed{t=4}$
b) $s(4)=80m$

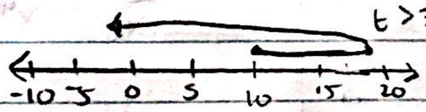
9. $s(t)=8-7t+t^2$
a) $s'(t)=-7+2t$
$s'(5)=3\ m/s$
b) $s''(t)=2$
$s''(5)=2\ m/s^2$

10. $s(t)=t^{5/2}(7-t)$
$s(t)=7t^{5/2}-t^{7/2}$
a) $s'(t)=\frac{35}{2}t^{3/2}-\frac{7}{2}t^{5/2}$
$s''(t)=\frac{105}{4}t^{1/2}-\frac{35}{4}t^{3/2}$
b) $\frac{35}{2}t^{3/2}-\frac{7}{2}t^{5/2}=0$
$\frac{35}{2}t^{3/2}=\frac{7}{2}t^{5/2}$
$5t^{3/2}=t^{5/2}$
$\boxed{5s=t}$
c) $\boxed{t=5s}$
d) $\frac{105}{4}t^{1/2}=\frac{35}{4}t^{3/2}$
$a(t)=0 \rightarrow \boxed{3=t}$
$s''(1)=(+)$
∴ $v(t)\ a(t)$ $0<t<3$

11. $h(t)=-5t^2+25t$
a) $h'(t)=-10t+25$
$h'(0)=25\ m/s$
b) $-10t+25=0$
$t=2.5s$
$h(2.5)=31.25m$
c) $-5t(t-5)=0$
$t=0,\ t=5$
$h'(5)=-10(5)+25$
$=-25\ m/s$

12. $s(t)=6t^2+2t$
a) $s'(t)=12t+2$
$s''(t)=12$
$s'(8)=98\ m/s$
$s''(8)=12\ m/s^2$
b) $6t^2+2t=60$
$2(3t^2+t-30)=0$
$2(3t+10)(t-3)=0$
$\cancel{t=-10/3}$ $t=3$
$s'(3)=12(3)+2$
$=38\ m/s$

13a) $s(t)=10+6t-t^2$ $s(0)=10$
$s'(t)=6-2t$ $\boxed{t=3}$
$=-2(t-3)$
$v>0$ when $t>3$



$s(3)=19m$
$v<0$
$0<t<3$

e) $t^{5/2}(7-t)=0$
$t=0,\ \boxed{t=7}$

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14. $s(t) = t^5 - 10t^2$
$s'(t) = 5t^4 - 20t$
$s''(t) = 20t^3 - 20$
$20t^3 - 20 = 0$
$t^3 = 1$
$t = 1\ s$
$s'(1) = 5 - 20$
$= -15$
$\therefore$ moving away from the origin (initial position is $(0,0)$)

15. $s(t) = kt^2 + (6k^2 - 10k)t + 2k$
a) $s'(t) = 2kt + 6k^2 - 10k$
$s''(t) = 2k$ ← constant, so acceleration is constant
b) $2kt + 6k^2 - 10k = 0$
$2k(t + 3k - 5) = 0$
$t = -3k + 5$
$s(-3k+5) = k(-3k+5)^2 + (6k^2-10k)(-3k+5) + 2k$
$= k(9k^2 - 30k + 25) - 18k^3 + 30k^2 + 30k^2 - 50k + 2k$
$= 9k^3 - 30k^2 + 25k - 18k^3 + 60k^2 - 48k$
$= -9k^3 + 30k^2 - 23k$

3.2 Maximum and Minimum Values on an Interval

1a) $y = x^3 - 5x^2 + 10,\ -5 \le x \le 5$
Polynomial function - continuous
$\therefore$ We can use it.

b) $y = \frac{3x}{x-2},\ -1 \le x \le 3$
Rational fcn, interval includes asymptote.
$\therefore$ We cannot use it

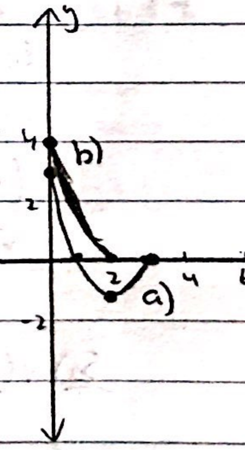
c) $y = \frac{x}{x^2-4},\ x \in [0,5]$
Interval includes $x = 2$ (asymptote)
so the algorithm cannot be used.

d) $y = \frac{x^2-1}{x+3},\ x \in [-2,3]$
Continuous on the interval,
so it can be used.

2a) min: $y = -12$ max: $y = 8$
b) min: $y = -5$ max: $y = 30$
c) min: $y = -100$ max: $y = 100$
d) min: $y = -20$ max: $y = 30$

3a) $f(x) = x^2 - 4x + 3 \quad 0 \le x \le 3$
max → $f(0) = 3 \quad f(3) = 0$
$f'(x) = 2x - 4 \quad f(2) = -1$ ← min
$2x - 4 = 0$
$x = 2$

b) $f(x) = (x-2)^2,\ 0 \le x \le 2$
$f(0) = 4$ ← max
$f(2) = 0$ ← min
$f'(x) = 2(x-2)$
$2x = 4 \quad f(2) = 0$
$x = 2$



3c) $f(x) = x^3 - 3x^2, \; -1 < x < 3$

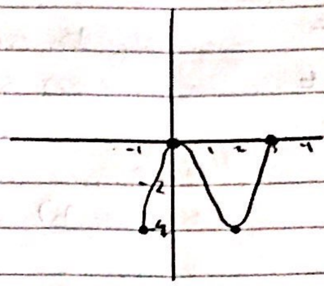
$f(-1) = -4 \qquad f(3) = 0$ ✓max

$f(0) = 0 \qquad f(2) = -4$ ✓min

$f'(x) = 3x^2 - 6x$

$3x(x-2) = 0$

$x = 0 \quad x = 2$



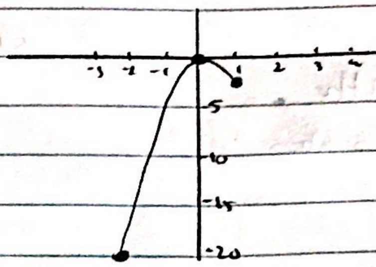
d) $f(x) = x^3 - 3x^2, \; x \in [-2, 1]$

$f(-2) = -20$ ✓min $\quad f(1) = -2$

$f(0) = 0$ ✓max $\quad$ ~~$f(2) = -4$~~

$f'(x) = 3x^2 - 6x$

$3x(x-2) = 0$



e) $f(x) = 2x^3 - 3x^2 - 12x + 1, \; x \in [-2, 0)$

min → $f(-2) = -3 \qquad f(0) = 1$

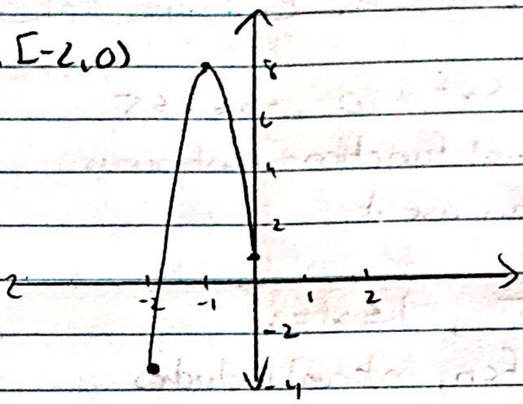
max → $f(-1) = 8$

$f'(x) = 6x^2 - 6x - 12$

$6(x^2 - x - 2) = 0$

$6(x-2)(x+1) = 0$

~~$x = 2$~~, $x = -1$



f) $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x, \; x \in [0, 4]$

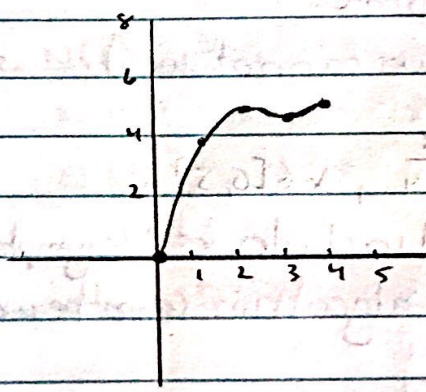
$f(0) = 0$ ✓min $\quad f(4) = \frac{16}{3}$ ← max

$f(2) = \frac{14}{3} \qquad f(3) = 9/2$

$f'(x) = x^2 - 5x + 6$

$(x-3)(x-2) = 0$

$x = 3, \; x = 2$



5a) $v(t) = \dfrac{4t^2}{4+t^3} \quad 1 < t < 4$

$v(1) = 4/5$ m/s ✓min $\qquad v(4) = 16/17$

$v(0) = 0$

$v(2) = 4/3$ ✓max

$v'(t) = \dfrac{8t(4+t^3) - 4t^2(3t^2)}{(4+t^3)^2}$

$= \dfrac{-4t^4 + 32t}{(4+t^3)^2}$

$-4t^4 + 32t = 0$

$-4t(t^3 - 8) = 0$

$-4t(t-2)(t^2+2t+4) = 0$

5b) $v(t) = \frac{4t^2}{1+t^2}, t \geq 0$

$v'(t) = \frac{8t(1+t^2) - 2t(4t^2)}{(1+t^2)^2}$

$= \frac{8t}{(1+t^2)^2}$

$t = 0$

$v(0) = 0$ ← min

$\lim_{t\to\infty} v(t) = \frac{t^2(4)}{t^2(\frac{1}{t^2}+1)}$

$= 4$

$v(t)$ approaches 4 as t approaches ∞

4a) $f(x) = x + \frac{4}{x}, 1 \leq x \leq 10$

$f'(x) = 1 - \frac{4}{x^2}$

$\frac{4}{x^2} = 1$

$4 = x^2$

$\pm 2 = x$

$f(1) = 5$

$f(10) = 10\frac{2}{5}$ ← max

$f(2) = 4$ ← min

b) $f(x) = 4\sqrt{x} - x, x \in [2,9]$

$f'(x) = 2x^{-1/2} - 1$

$2/\sqrt{x} - 1 = 0$

$\frac{2}{\sqrt{x}} = 1$

$4 = x$

$f(2) = 4\sqrt{2} - 2$

$f(9) = 3$ ← min

$f(4) = 4$ ← max

c) $f(x) = \frac{1}{x^2-2x+2}, 0 \leq x \leq 2$

$f'(x) = \frac{-(2x-2)}{(x^2-2x+2)^2}$

$-2(x-1) = 0$

$x = 1$

$f(0) = \frac{1}{2}$ ← min

$f(2) = \frac{1}{2}$ ← min

$f(1) = 1$ ← max

e) $f(x) = \frac{4x}{x^2+1}, -2 \leq x \leq 4$

$f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2}$

$= \frac{-4x^2+4}{(x^2+1)^2}$

$-4(x^2-1) = 0$

$x^2 = 1$

$x = \pm 1$

$f(-2) = -\frac{8}{5}$

$f(4) = \frac{16}{17}$

$f(-1) = -2$ ← min

$f(1) = 2$ ← max.

f) Same function, but $x = \pm 1$ are not in the interval

$f(2) = 8/5$ ← max.

$f(4) = 16/17$ ← min

6. $N(t) = 30t^2 - 240t + 500, 0 \leq t \leq 7$

$N(0) = 500$ $N(7) = 290$

$N'(t) = 60t - 240$

$60t = 240$

$t = 4$

min

$N(4) = 20$ bacteria

d) $f(x) = 3x^4 - 4x^3 - 36x^2 + 20 \; x \in [-3, 4]$

$f'(x) = 12x^3 - 12x^2 - 72x$

$12x(x^2 - x - 6) = 0$

$12x(x-3)(x+2) = 0$

$x = 0, x = 3, x = -2$

$f(-3) = 47$ ← max.

$f(-2) = -44$

$f(0) = 20$

$f(3) = -169$ ← min

$f(4) = -44$

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7. $\mathcal{E}(v) = \frac{1600v}{v^2+6400}$, $0 \le v \le 100$

$\mathcal{E}(0) = 0$ $\quad \mathcal{E}(100) = \frac{400}{41}$

$\mathcal{E}(80) = 10$ ← max at 80 km/h

a) $\mathcal{E}'(v) = \frac{1600(v^2+6400) - 2v(1600v)}{(v^2+6400)^2}$

$= \frac{-1600v^2 + 10\,240\,000}{(v^2+6400)^2}$

$v^2 = 6400$

$v = \pm 80$

b) $0 \le v \le 50$ ← max 50 km/h

$\mathcal{E}(50) = \frac{800}{89}$

c)d) ↑ from 0 to 80 km/h, ↓ 80 to 100 km/h

8. $C(t) = \frac{0.1t}{(t+3)^2}$, $1 \le t \le 6$

min @ $t = 1$.

$C(1) = 1/160$ $\quad C(6) = \frac{1}{135}$

$C'(t) = \frac{0.1(t+3)^2 - 0.1t(2(t+3))}{(t+3)^4}$

$= \frac{0.1(t^2+6t+9) - 0.2t^2 - 0.6t}{(t+3)^4}$

$= \frac{-0.1t^2 + 0.9}{(t+3)^4}$

$0.1t^2 = 0.9$

$t^2 = 9$

$t = \pm 3$

max @ $t = 3$

$C(3) = \frac{1}{120}$

9. $P(t) = 2t + \frac{1}{(162t+1)}$, $0 \le t \le 1$

$P(0) = 1$ $\quad P(1) = 2\frac{1}{163}$

$P'(t) = 2 - \frac{162}{(162t+1)^2}$

$2(162t+1)^2 = 162$

$(162t+1)^2 = 81$

$162t+1 = \pm 9$

$162t = 8$ $\quad 162t = -10$

$t = 4/81$ $\quad t = -5/81$

$P(4/81) = \frac{8}{81} + \frac{1}{9}$

$= 17/81$

min at $t = 4/81$

10. $30 \le x \le 120$

$r(x) = \frac{1}{4}\left(\frac{4900}{x} + x\right)$

$r(30) = 48\frac{1}{3}$ $\quad r(120) = 40\frac{5}{24}$

$r(x) = \frac{4900}{4x} + \frac{1}{4}x$

$r'(x) = -\frac{4(4900)}{16x^2} + \frac{1}{4}$

$= \frac{-1225}{x^2} + \frac{1}{4}$

$4900 = x^2$ $\quad r(70) = \frac{70}{2}$

$70 = x$

∴ min cost at 70 km/h

$\frac{70}{2}(2) = 70\,L \times \$1.15/L$

$= \$80.50$ ← cost

11. $f(x) = 0.001x^3 - 0.12x^2 + 3.6x + 10$, $0 \le x \le$

$f'(x) = 0.003x^2 - 0.24x + 3.6$

$0.003(x^2 - 80x + 1200) = 0$

$0.003(x-60)(x-20) = 0$

$f(0) = 10$ $\quad f(60) = 10$ ← min height

$f(75) = 26.875$ $\quad f(20) = 42$ ← max height

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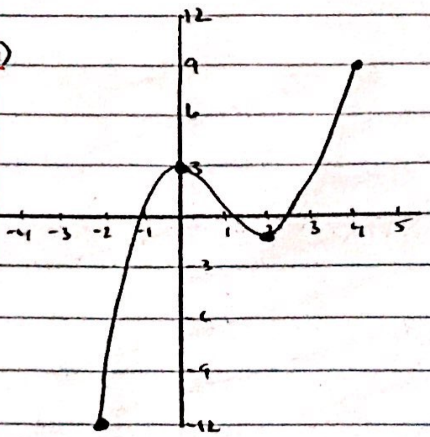
12a)



b) $D = \{x \in \mathbb{R} \mid -2 \le x \le 4\}$

c) $\uparrow x \in (-2,0), (2,4)$

$\downarrow x \in (0,2)$

13. Endpoint and turning point

14. $C(x) = 3000 + 9x + 0.05x^2, \quad 1 \le x \le 300$

$U(x) = \frac{3000}{x} + 9 + 0.05x$

$U(1) = 3009.05 \quad U(300) = 34$

$U'(x) = -\frac{3000}{x^2} + 0.05$

$0.05x^2 = 3000$

$x^2 = 60000$

$x = 244.9$

min @ 245 units

$U(245) = 33.5$