Calculus Appendix: Related Rates

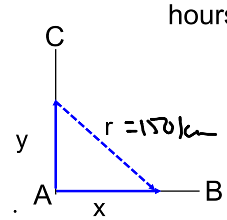
Now that you know what implicit differentiation is, we can apply derivatives to real world problems. Often **two quantities change in relation to each other.**

For example, we solved lots of motion problems where one thing (usually a train) left from a fixed point, and another arrived at it. In real life, this happens sometimes, but we also will encounter two things moving relative to one another (what if two trains leave from the same spot and go in different directions??).

Also, when you are examining the rate of change of volume in a container that is not cylindrical you need to be aware that the rate at which the volume changes is dependent on the radius at that particular height (we did lots with prisms and cylinders where the dimensions are constant, but what if we are filling a cone with water??).

In order to solve these problems, we need to set up equations using the chain rule and use implicit differentiation, as we have two variables that are changing.

Example 1: Two cars start from point A and travel along perpendicular roads AB and AC (as shown below). The first car drives at a speed of 45 km/h along AB and the second travels at a speed of 40 km/h along AC. If the second car starts driving 1 hour before the first, at what rate are their cars separating 3 hours after the first car leaves?




*Let x be the distance that the first car has travelled along AB and y be the distance that the second car has travelled along AC.*

Given

$x = 45\text{km/h} \times 2\text{h}$

$x = 90\text{km}$

$y = 40\text{km/h} \times 3\text{h}$

$y = 120\text{km}$

$r^2 = 90^2 + 120^2$

$r = 150\text{ km}$

$\frac{dx}{dt} = 45 \quad \frac{dy}{dt} = 40$

$\boxed{\text{We need } \frac{dr}{dt}}$

$x^2 + y^2 = r^2$ ← differentiate w.r.t 't'.

$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2r \cdot \frac{dr}{dt}$

$x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = r \cdot \frac{dr}{dt}$

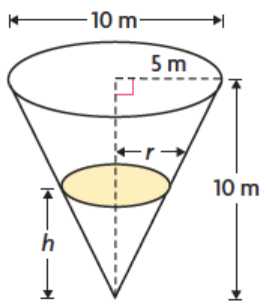
$90(45) + 120(40) = 150 \frac{dr}{dt}$

$\frac{8850}{150} = \frac{dr}{dt}$

$59 = \frac{dr}{dt}$

∴ The cars are separating at a rate of 59km/h.

Example 2: Water is pouring into an inverted right circular cone at a rate of $\pi$ m$^3$/min. The height and the diameter of the base of the cone are both 10 m. How fast is the water level rising when the depth of the water is 8 m?




$V = \frac{1}{3}\pi r^2 h$ (write in terms of h)

$V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h$

$V = \frac{1}{12}\pi h^3$ ← differentiate wrt time.

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \cdot \frac{dh}{dt}$

We want $\frac{dh}{dt}$

Given: $\frac{dV}{dt} = \pi \frac{m^3}{min}$

$h = d = 10$, $r = \frac{1}{2}h$

$h = 8$, $r = 4$

$\pi = \frac{1}{4}\pi(8)^2 \cdot \frac{dh}{dt}$

$\frac{1}{16} = \frac{dh}{dt}$

∴ The water is rising at $\frac{1}{16}$ m/min.

Example 3: A bicycle has tires with radius 35 cm. If the bicycle moves forward at a speed of 6 m/s, at what rate do the tires turn?

Let x be distance travelled, n → number of revolutions

$x = C \cdot n$

$x = 2\pi r n$

$x = 0.7\pi n$

$x = 0.7\pi n$ ← diff wrt time

We want: $\frac{dn}{dt}$

$\frac{dx}{dt} = 0.7\pi \frac{dn}{dt}$

$r = 0.35$ m

$\frac{dx}{dt} = 6$ m/s

$\frac{6}{0.7\pi} = \frac{dn}{dt}$

$2.73 = \frac{dn}{dt}$

∴ The rate is 2.73 rev/s when speed is 6 m/s.