3.3 Optimization Problems

1. $A = lw$, $0 \le x \le 50$

$P = 2l + 2w$ $\quad$ $A(w) = -w^2 + 50w$

$l = \frac{100 - 2w}{2}$ $\quad$ $A'(w) = -2w + 50$

$-2w + 50 = 0$

$l = 50 - w$ $\quad$ $w = 25m$

$\therefore$ The dimensions are 25m x 25m.

2. It produces a square.

3. $A = lw$ $\quad$ $A(w) = (600 - 2w)w$

$P = 2w + l$ $\quad$ $A'(w) = -4w + 600$

$0 < w < 300$ $\quad$ $l = 600 - 2w$ $\quad$ $-4(w - 150) = 0$

$w = 150,\ l = 300$

$\therefore$ The dimensions are 300m x 150m.

4. $V(h) = (100 - 2h)(40 - 2h)(h)$, $0 < h < 20$

$= 4000h - 200h^2 - 80h^2 + 4h^3$

$V'(h) = 4000 - 400h - 160h + 12h^2$

$= 12h^2 - 560h + 4000$

$0 = 4(3h^2 - 140h + 1000)$

$h = \frac{140 \pm \sqrt{19600 - 12000}}{6}$

$= \frac{140 \pm \sqrt{7600}}{6}$

$= \frac{140 \pm 10\sqrt{76}}{6}$

$= \frac{70 \pm 5\sqrt{76}}{3}$

$h = 38$ too big!

$\boxed{h = 8.8cm}$

$\boxed{l = 82.4cm}$

$\boxed{w = 22.4cm}$

5. $A = lw$ $\quad$ $0 < w < 220$

$P = 2l + 2w$

$l = 220 - w$

$A(w) = -w^2 + 220w$

$A'(w) = -2w + 220$

$2w = 220$

$w = 110$ $\quad$ $\therefore$ The dimensions are 110m x 110m

6. $2l + 2w = P$

$A = lw$

$lw = 64$ $\quad$ $0 < x < 64$

$w = \frac{64}{l}$

$P(l) = 2l + \frac{128}{l}$

$P'(l) = 2 - \frac{128}{l^2}$

$2l^2 = 128$

$l^2 = 64$

$l = 8$ m x 8m

7. $P = 4x + 3y$ $\quad$ $0 < y < \frac{1000}{3}$

$4x + 3y = 1000$

$x = -\frac{3}{4}y + 250$

$A(y) = y\left(-\frac{3}{4}y + 250\right)$

$A'(y) = -\frac{3}{2}y + 250$

$\frac{3}{2}y = 250$

$\boxed{y = \frac{500}{3}m}$

$4x + 500 = 1000$

$4x = 500$

$\boxed{x = 125m}$

$<x<12$

8. $V = x^2 y$ $\qquad SA = 3xy + x^2$

$y = \frac{144}{x^2}$ $\qquad SA(x) = 3x\left(\frac{144}{x^2}\right) + x^2$

$= \frac{432}{x} + x^2$

$SA'(x) = \frac{-432}{x^2} + 2x$

$\frac{432}{x^2} = 2x$ $\qquad (6^2)y = 144$

$432 = 2x^3$ $\qquad y = 4$

$216 = x^3$

$6 = x$

∴ The dimensions are 4m×6m×6m.

9. $w^2 h = 1000$ $\qquad SA = 2w^2 + 4wh$

$h = \frac{1000}{w^2}$ $\qquad 2 < w < \sqrt{500}$

$2 < h < 250$

$SA(w) = 2w^2 + 4w\left(\frac{1000}{w^2}\right)$

$= 2w^2 + \frac{4000}{w}$ $\qquad SA(2) = 8 + 2000 = 2008$

$SA'(w) = 4w - \frac{4000}{w^2}$ $\qquad SA(\sqrt{500}) = 1178.9$

$\boxed{SA(10) = 200 + 400 = 600cm^2}$

$\frac{4w^3 - 4000}{w^2} = 0$

$w^3 = 1000$

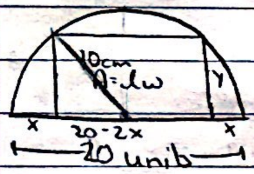
$w = 10cm$

$(10)^2 h = 1000$

$h = 10 cm$

∴ The dimensions are 10×10×10.

10.



$A_{semicircle} = \frac{1}{2}\pi r^2$ $\qquad A_{\square} = (20-2x)y$

$= 50\pi$ sq units

$y^2 + (10-x)^2 = 100$

$y^2 + 100 - 20x + x^2 = 100$

$y^2 = 20x - x^2$

$y = \sqrt{x(20-x)}$

$A(x) = (20-2x)(20x-x^2)^{\frac{1}{2}}$

$A'(x) = -2(20x-x^2)^{\frac{1}{2}} + \frac{1}{2}(20x-x^2)^{-\frac{1}{2}}(20-2x)(20-2x)$

$= -2(20x-x^2)^{\frac{1}{2}} + \frac{(20-2x)^2}{2(20x-x^2)^{\frac{1}{2}}}$

$0 = \frac{-4(20x-x^2) + (20-2x)^2}{2(20x-x^2)^{\frac{1}{2}}}$ $\qquad \boxed{0 < x < 20}$

$0 = \frac{-80x + 4x^2 + 400 - 80x + 4x^2}{2(20x-x^2)^{\frac{1}{2}}}$

$0 = 4x^2 - 80x + 200$

$0 = x^2 - 20x + 50$

$x = \frac{20 \pm \sqrt{20^2 - 4(1)(50)}}{2}$

$= \frac{20 \pm 10\sqrt{2}}{2}$

$= 10 \pm 5\sqrt{2}$

$x = 17.07$ $\qquad x = 2.93$

11a) $V=\pi r^2 h$ $\quad$ $SA = 2\pi r^2 + 2\pi r h$

$\pi r^2 h = 1000$ $\quad$ $SA(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$

$h > 4cm$ $\quad$ $h = \frac{1000}{\pi r^2}$

$= 2\pi r^2 + \frac{2000}{r}$

$r > 3cm$

$h:d = 4:6$

$= 2:3$

$SA'(r) = 4\pi r - \frac{2000}{r^2}$

$0 = \frac{4\pi r^3 - 2000}{r^2}$

$2000 = 4\pi r^3$

$\sqrt[3]{\frac{500}{\pi}} = r$

$5.42^{cm} = r$

$h = \frac{1000}{\pi(5.42)^2}$

$= 10.83 cm$

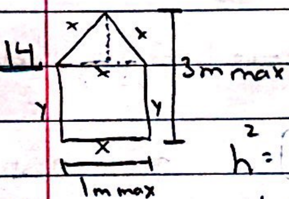
b) $h:d$

$= 10.83 : 10.84$

$= 1:1$

$\therefore$ It works!

14.



$h^2 = x^2 - \left(\frac{1}{2}x\right)^2$

$h = \sqrt{x^2 - \frac{1}{4}x^2}$

$= \sqrt{\frac{3}{4}x^2}$

$= \frac{\sqrt{3}}{2}x$

a) $4x + 2y = 6$

$y = -2x + 3$

$A = xy + \frac{1}{2}\left(\frac{\sqrt{3}}{2}x^2\right)$

$A(x) = -2x^2 + 3x + \frac{\sqrt{3}}{4}x^2$

$A'(x) = -4x + 3 + \frac{\sqrt{3}}{2}x$

$0 = -8x + 6 + \sqrt{3}x$

$-6 = x(-8 + \sqrt{3})$

$0.96m = x$

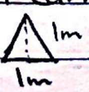
$y = -2(0.96) + 3$

$= 1.08 cm$

$A = \frac{0.96 \times 1.08}{2}$

$= 0.5184 m^2$

b) No → the width can't be more than 1m, so:

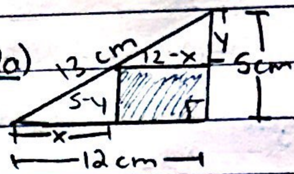


$h = 1^2 - \left(\frac{1}{2}\right)^2$

$= \frac{3}{4} m$, $A = \frac{3}{4} \div 2$

$= 3/8\ m^2$

12a)



Similar triangles:

$\frac{12-x}{y} = \frac{12}{5}$

$\frac{60-5x}{12} = y$

$A = (12-x)(5-y)$

$A(x) = (12-x)\left(5 - \frac{60-5x}{12}\right)$

$= (12-x)\left(\frac{-5x}{12}\right)$

$A'(x) = \frac{-60x + 5x^2}{12}$

$= \frac{-5x}{12} + \frac{5x^2}{12}$

$A'(x) = -\frac{5}{12} + \frac{5}{6}x$

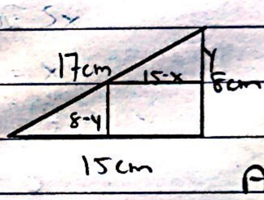
$\frac{5}{12} = \frac{5}{6}x$

$6 = x$

$y = 5/2$

$\boxed{A = 15 cm^2}$

b)



$\frac{15-x}{y} = \frac{15}{8}$

$8(15-x) = 15y$

$\frac{8(15-x)}{15} = y$

$A = (8-y)(15-x)$

$A(x) = \left(8 - \frac{120-8x}{15}\right)(15-x)$

$= \left(-\frac{8x}{15}\right)(15-x)$

$= \frac{-120x + 8x^2}{15}$

$= -8x + \frac{8}{15}x^2$

$A'(x) = -8 + \frac{16}{15}x$

$8 = \frac{16}{15}x$

$7.5 = x$ $\quad$ $y = 4$

$A = 30 cm^2$

c) The length/width of the rectangle should be $\frac{1}{2}$ that of the adjac. triangle side.

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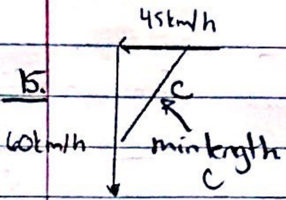
15.



After 1 hour, they are 60km apart. Initially they were 45km apart. $45 \le d \le 60$ $\quad 0 < t < 1$

Let t be time in hours, and d be distance from station.

Train 1: $d = 60t$ $\quad$ Train 2: $d = 45 - 45t$

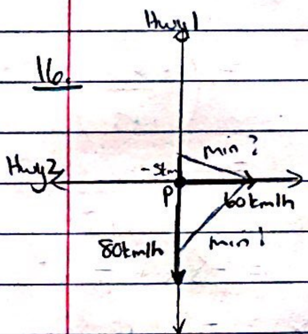
$$c^2 = (60t)^2 + (45-45t)^2$$
$$c^2 = 3600t^2 + 2025 - 4050t + 2025t^2$$
$$c^2 = 5625t^2 - 4050t + 2025$$
$$c(t) = \left[225(25t^2 - 18t + 9)\right]^{1/2}$$
$$= 15(25t^2 - 18t + 9)^{1/2}$$
$$c'(t) = \frac{15}{2}(25t^2 - 18t + 9)^{-1/2}(50t - 18)$$

$$0 = \frac{15(50t-18)}{2\sqrt{(25t^2-18t+9)}}$$

$$50t = 18$$

$t = 0.36$ hours after the train left the station.

16.



Let t be time since car 1 passed P, and d be distance from P.

$d = 60t$ $\qquad d = -5 + 80t$

$$c^2 = (60t)^2 + (-5 + 80t)^2$$
$$c^2 = 3600t^2 + 25 - 800t + 6400t^2$$
$$c(t) = (10000t^2 - 800t + 25)^{1/2}$$
$$= 5(400t^2 - 32t + 1)^{1/2}$$
$$c'(t) = \frac{5}{2}(400t^2 - 32t + 1)^{-1/2}(800t - 32)$$

$$0 = \frac{5(800t-32)}{2\sqrt{400t^2-32t+1}}$$

$$800t = 32$$

$t = 0.04$ h (2.4 minutes)

$d = 60(0.04)$
$= 2.4$ km

$d = -5 + 80(0.04)$
$= 1.8$ km.

$c = \sqrt{1.8^2 + 2.4^2}$
$= 3$ km

$\therefore$ They are closest at 1:02 pm, and the distance is 3km.

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3.4 Optimization Problems in Economics + Science

1a) $C(625) = 75(25-10)$ aug. cost:
$= \$1125$ $\$1.80/L$

b) $C'(x) = \frac{75}{2}x^{-1/2}$

$= \frac{75}{2\sqrt{x}}$

$C'(1225) = \frac{75}{2(35)}$

$= {}^{15}/_{14}$

$= \$1.07/L$

c) $0.5 = \frac{75}{2\sqrt{x}}$

$\sqrt{x} = 75$

$x = 5625L$

2. $N(t) = 20t - t^2$

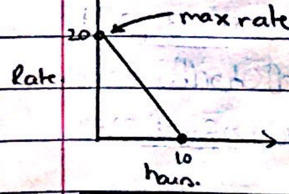
a) $N(2) = 36$, $N(3) = 51$

$m = \frac{51-36}{1} = 15$ terms

b) $N'(t) = 20 - 2t$

$N'(2) = 16$ words/hour

c) $N'(t) = -2t + 20$



3. $L(t) = \frac{6t}{t^2+2t+1}$

a) $L'(t) = \frac{6(t^2+2t+1) - 6t(2t+2)}{(t^2+2t+1)^2}$

$0 = \frac{6t^2+12t+6-12t^2-12t}{(t^2+2t+1)^2}$

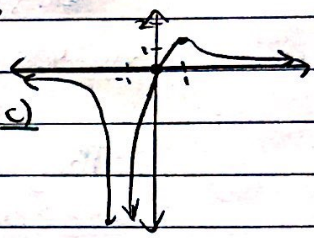
$0 = -6t^2 + 6$

$\boxed{t = 1}$

b) $L(1) = \frac{6}{1+2+1}$

$= {}^{3}/_{2}$

c)



d) The level decreases (approaches zero)

e) Approaching zero.

4. $C = 4000 + \frac{h}{15} + \frac{15000000}{h}$, $1000 \leq h \leq 20000$

$C' = \frac{1}{15} + \frac{-15000000}{h^2}$

$0 = \frac{h^2 - 225000000}{15h^2}$

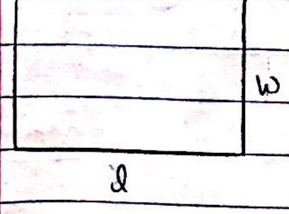
$h^2 = 225000000$

$h = 15000$ m

$C = 4000 + \frac{15000}{15} + \frac{15000000}{15000}$

$= \$6000/h$

5.



$2l = x \rightarrow$ m of \$6/m fence

$2w = y \rightarrow$ m of \$9/m fence

Cost $= 6x + 9y = 9000$

$y = -\frac{2}{3}x + 1000$

$A = lw$

$= (\frac{1}{2}x)(\frac{1}{2}y)$

$= \frac{1}{4}xy$

$2l = x$ $2w = y$

$l = 375$m $w = 250$

$A(x) = -\frac{1}{6}x^2 + 250x$

$A'(x) = -\frac{1}{3}x + 250$

$x = 750$ m

$9y = 9000 - 6(750)$

$y = 500$m

$\therefore$ The dimensions are 375m x 250m

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$x \to$ # of price ↑s

$y \to$ # of units rented

$\boxed{y = 50 - x}$

$0 \le x \le 50$

$P(x) \ge 0$

6. $P = R - C$

$R(x) = (50-x)(900+25x)$

$C(y) = 75y$

$P(x) = (50-x)(900+25x) - 75(50-x)$

$= (50-x)[(900+25x) - 75]$

$= (50-x)(825+25x)$

$P'(x) = -(825+25x) + 25(50-x)$

$= -825 - 25x + 1250 - 25x$

$= -50x + 425$

$50x - 425 = 0$

$x = 8.5$

∴ They maximize profit after 8 or 9 increases ($1100 or $1125)

$t \to$ time in h.

$x \to$ speed in knots

knot → 1 M/h

$d = st$

$500 = xt$

$t = \frac{500}{x}$

8. $Cost = t\left(\frac{x^3}{2} + 216\right)$

$C(x) = \left(\frac{500}{x}\right)\left(\frac{x^3}{2} + 216\right)$

$= 250x^2 + \frac{108000}{x}$

$= \frac{250x^3 + 108000}{x}$

$C'(x) = \frac{(750x^2)(x) - (250x^3 + 108000)}{x^2}$

$= \frac{750x^3 - 250x^3 - 108000}{x^2}$

$= \frac{500x^3 - 108000}{x^2}$

$500x^3 = 108000$

$x^3 = 216$

$x = 6$

∴ 6 knots is the best speed.

Max people → 15000

$R(x) \ge \$130000$

7. $x \to$ # of price increases

$R(x) = (20 + 0.5x)(10000 - 200x)$

$R'(x) = 0.5(10000 - 200x) - 200(20 + 0.5x)$

$= 5000 - 100x - 4000 - 100x$

$200x - 1000 = 0$

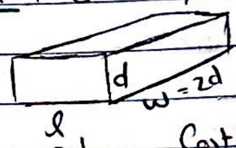
$x = 5$

$R(5) = (20 + 2.50)(10000 - 1000)$

$= \$202500$

∴ $22.50 maximizes the revenue.

9. $1 \le d \le 22$, d is depth in meters



$w = 2d$

$V = 20000\ m^3$

$SA = A_{base} + 4A_{side} + A_{top}$

$Cost = 40A_{base} + 2(100A_{side}) + 200A_{roof}$

$lwd = 20000$

$l(2d)d = 20000$

$2d^2 l = 20000$ $\quad SA = 2dl + 2dw + 2lw$

$l = \frac{20000}{2d^2}$

$l = \frac{10000}{d^2}$

$C = 40lw + 200lw + 100(2ld + 2dw)$

$C(d) = 40\left(\frac{10000}{d^2}\right)(2d) + 200\left(\frac{10000}{d^2}\right)(2d) + 100\left(2d\left(\frac{10000}{d^2}\right) + 2d(2d)\right)$

$= \frac{800000}{d} + \frac{4000000}{d} + 100\left(\frac{20000}{d} + 4d^2\right)$

$= \frac{4800000}{d} + \frac{2000000}{d} + 400d^2$

$= \frac{6800000 + 400d^3}{d}$

$C'(d) = \frac{d(1200d^2) - (6800000 + 400d^3)}{d^2}$

∴ $800d^3 = 6800000$

$d = \sqrt[3]{8500} = 20.4m$

$d = 20.4m$

$w = 40.8m$

$l = 24.03m$

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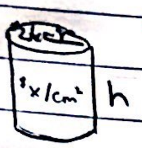
sides ↓ top + bottom ↓ doubled to reflect ↓ doubling the cost.



10. $V = \pi r^2 h$

$\pi r^2 h = 1000$

$h = \frac{1000}{\pi r^2}$

$SA = 2\pi r h + 2\pi r^2$

$SA(r) = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2$

$= \frac{2000}{r} + 2\pi r^2$

$C(r) = \frac{2000}{r} + 4\pi r^2$

$C'(r) = -\frac{2000}{r^2} + 8\pi r$

$2000 = 8\pi r^3$

$\sqrt[3]{\frac{2000}{8\pi}} = r$

$4.3\,cm = r$ $h = 17.2\,cm$

11. $P(x) = R(x) - C(x)$

y → # of cakes

x → # of price changes.

$R(x) = (10 + 0.5x)(200 - 7x)$

$C = 6y$

$C(x) = 6(200 - 7x)$

$P(x) = (200 - 7x)[(10 + 0.5x) - 6]$

$= (200 - 7x)(4 + 0.5x)$

$= 800 + 100x - 28x - 3.5x^2$

$= 800 + 72x - 3.5x^2$

a) $P'(x) = 72 - 7x$

$7x = 72$

$x = 10.3$ price ↑'s. ($950)

∴ The optimum price is $15.00. (130 cakes)

b) $x \leq 5$ for P(x) from a) to hold true.

$P(5) = \$990$

If $x > 5$, $P(x) = (165 - 7x)(12.50 + 0.1x) - 7.5(165 - 7x)$

$= (165 - 7x)(5 - 0.1x)$

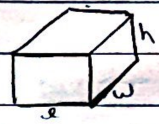
$= 825 - 51.5x + 0.7x^2$

$P'(x) = 1.4x - 51.5$

$x = 36.8$ Price = 16.20, Q = -93

∴ The max would occur at the initial condition, $12.50, $825 profit.

12.



$V = lwh$ ← $l = w$ (square)

$\frac{4000}{l^2} = h$

$\frac{\$20}{m^2} = \frac{m^2}{10000\,cm^2}$ $\$0.002/cm^2$

$Cost = 0.002(2l^2) + 0.003(4lh)$

$C(l) = 0.004l^2 + 0.012l\left(\frac{4000}{l^2}\right)$

$C'(l) = \frac{0.004l^3 + 48}{l}$

$C'(l) = \frac{l(0.012l^2) - (0.004l^3 + 48)}{l^2}$

$= \frac{0.008l^3 - 48}{l^2}$

$0.008l^3 = 48$

$l^3 = 6000$

$l = 18.2\,cm$

$\frac{4000}{(18.2)^2} = h$

$12.1 = h$

13. $R(x) = (90 - x)(50 + 5x)$

$R'(x) = 5(90 - x) + (-1)(50 + 5x)$

$= 450 - 5x - 50 - 5x$

$= -10x + 400$

$x = 40$

∴ $50 maximizes revenue.

14. $R(x) = (14000 + 800x)(75 - 5x)$

max people ↓ $0 \leq n \leq 20000$

$R'(x) = -5(14000 + 800x) + 800(75 - 5x)$

$= -70000 - 4000x + 60000 - 4000x$

$= -8000x - 10000$

$x = -1.25$

Price $= 75 + 5(1.25)$

$81.25

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price x quantity

15. Profit = revenue - cost

$P(x) = 1000x(2000 - 5x) - (15\,000\,000 + 1800\,000x + 75x^2)$

$P(x) = -5075x^2 - 200\,000x - 15\,000\,000$

$P'(x) = -10150x - 200\,000$

$x = 19.704$

$\therefore$ They need to produce 19704 units.

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