Calculus Appendix Part 1: Implicit Differentiation (Take 3 days to work through this)

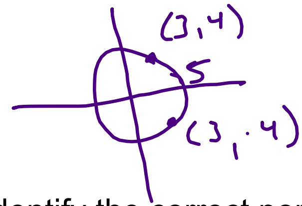
Until now, you have worked primarily with functions that are written so that y is explicitly defined in terms of x. Some examples are given below.

$$y = \sqrt{2x-5} \qquad f(x) = \frac{8x}{x^3-5} \qquad y = 2sinx$$

In the event that you saw a function defined in terms of two variables, until now you would just rearrange it to write it as a function of y in terms of x.

a) Rewrite $x^2 + y^2 = 25$ in terms of y.

$y = \pm\sqrt{25 - x^2}$



We can differentiate this function, as long as we identify the correct portion to use when we are asked to determine the slope of a tangent at a given point.

b) Determine the slope of the tangent to the curve in a) when x = 3. What issue have you run in to? How many answers do you need to find?

$$\frac{dy}{dx} = \frac{1}{2}(25-x^2)^{-1/2}(-2x)$$
$$= \frac{-x}{\sqrt{25-x^2}}$$

At $x = 3$:

$\frac{dy}{dx} = -\frac{3}{4}$ or $\frac{dy}{dx} = \frac{3}{4}$

Rather than rearranging to create an explicit relationship, we can use **implicit differentiation** to differentiate an equation with respect to x. To do this, we differentiate both sides of the equation with respect to x. We need to remember our derivative rules (product, quotient, chain, etc.) and Leibniz notation for the chain rule ($\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$) so that we do this properly.

Implicit Differentiation: Find the derivative of $x^2 + y^2 = 25$.

$$x^2 + y^2 = 25$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2yy' = -2x$$

$$y' = -\frac{x}{y}$$

$\frac{d}{dy} y^2 \cdot \frac{dy}{dx}$

At $(3,-4)$:

$y' = \frac{-3}{-4} = \frac{3}{4}$

At $(3,4)$:

$y' = -\frac{3}{4}$

Notes:

- Take the derivative of both sides with respect to x.
- Apply the chain rule when working with terms with 'y'.
- Isolate dy/dx to get the derivative.

The example that we did can be completed implicitly or explicitly. However, there are lots of functions that cannot be expressed explicitly.

For example, find the derivative of $2xy - y^3 = 4$.

$2y + 2xy' - 3y^2y' = 0$ product rule.

$\frac{dy}{dy} \cdot \frac{dy}{dx}$

$= 1y'$

$y'(2x - 3y^2) = -2y$

$y' = \frac{-2y}{2x - 3y^2}$

Because these derivatives are dependent on both x and y, we need to sub in values for both if we are finding the tangent slope (use the whole point of tangency).

Practice Problems:

1) Determine an equation of the tangent to the curve $x^3 + y^3 = 9$ at the point (2, 1).

$3x^2 + 3y^2y' = 0$

$y' = \frac{-x^2}{y^2}$

$y' = \frac{-4}{1}$

$m = -4$

$y = mx + b$

$1 = -4(2) + b$

$9 = b$

$y = -4x + 9$

2) Find the derivative of $x^3 + \underbrace{x^2y}_{\text{Product.}} + 4y^2 = 6$

$3x^2 + 2xy + x^2y' + 8yy' = 0$

$y'(x^2 + 8y) = -3x^2 - 2xy$

$y' = \frac{-(3x^2 + 2xy)}{x^2 + 8y}$

Questions to try: p. 564 #2 - 10, Handout #1 - Implicit Differentiation