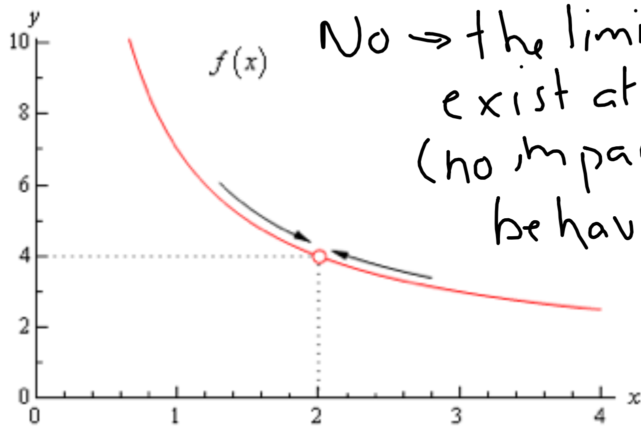
Friday, February 7, 2020

1.5 Properties of Limits

When we are thinking about the limit of a function, we are saying that the value of f(x) is getting closer and closer to the number L as x gets closer and closer to the number 'a' from either side.

Does the function need to be defined at 'a' for the limit as x approaches 'a' to exist? Think about this using the graph shown below.



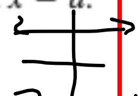
No → the limit can exist at a hole (no impact on the behaviour of the function)

There are a variety of properties of limits that allow us to simplify the process of finding the limit of f(x) as x appraoches 'a': (from p. 40)

Properties of Limits

For any real number a, suppose that f and g both have limits that exist at $x = a$.

1. $\lim_{x\to a} k = k$, for any constant k $\quad \lim_{x\to a} 3 = 3 \qquad f(x) = 3$



2. $\lim_{x\to a} x = a$ $\quad \lim_{x\to 2} x = 2$

$\lim_{x\to 3}\left[\frac{\sqrt{x+1}-2}{x} + \frac{\frac{1}{2+x}-2}{(x-5)}\right]$ ← split, then simp.

3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
4. $\lim_{x\to a}[cf(x)] = c[\lim_{x\to a} f(x)]$, for any constant c
5. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$
6. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$
7. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, for any rational number n

These properties give us more tools to use when we are trying to simpify complex functions and determine their limiting values.

Using Properties of Limits and Direct Substitution

Sometimes we can apply properties of limits and then simply plug in the value of 'a' to solve.

Example: Evaluate $\lim_{x\to5}\sqrt{\frac{x^2}{x-1}}$ by applying the properties of limits.

$$= \lim_{x\to5}\left(\frac{x^2}{x-1}\right)^{1/2}$$

$$= \left(\lim_{x\to5}\frac{x^2}{x-1}\right)^{1/2}$$

$$= \left(\frac{25}{4}\right)^{1/2} = \frac{5}{2}$$

Strategies to Use when Direct Substitution Fails

Sometimes we plug in our 'a' value and end up with $\frac{0}{0}$. We saw this often in sections 1.2 and 1.3 when we were looking for tangent slopes. This is called the indeterminate form of a function for reasons previously discussed. When this happens, we look for an equivalent function that works for all values except for x = a.

Option 1: Factor

When we have a polynomial function over a polynomial function, we can often eliminate our denominator (or at least the part that is producing zero) by factoring. What would this common factor in the numerator and denominator produce on a graph?

A hole.

Example: Evaluate $\lim_{x\to4}\frac{x^2-16}{x-4}$

$$= \lim_{x\to4}\frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}}$$

$$= \lim_{x\to4}(x+4)$$

$$= 4+4$$

$$= 8$$

Option 2: Rationalize

When you have a radical expression in your numerator, it is often helpful to rationalize the numerator to produce an expression that does not result in the indeterminate form.

Example: Evaluate $\lim_{x\to0}\frac{\sqrt{x+1}-1}{x} \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$$= \lim_{x\to0}\frac{x+1-1}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to0}\frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}$$

$$= \lim_{x\to0}\frac{1}{\sqrt{x+1}+1}$$

$$= \frac{1}{2}$$

Option 3: Substitution

Sometimes the expression that we are being asked to find the limit of is just ugly. When this happens, it is helpful to replace x with a different variable (u), and then solve for u in terms of x. This is easiest to explain through an example.

ugliest part!

Example: Evaluate $\lim_{x\to 0} \frac{(x+8)^{\frac{1}{3}}-2}{x}$

$u=(x+8)^{\frac{1}{3}}$

$u^3=x+8$

$u^3-8=x$

$u=(x+8)^{1/3}$

Let $x=0$

$u=2$

$= \lim_{u\to 2} \frac{u-2}{u^3-8}$

$= \lim_{u\to 2} \frac{\cancel{u-2}}{(\cancel{u-2})(u^2+2u+4)}$

$= \lim_{u\to 2} \frac{1}{u^2+2u+4}$

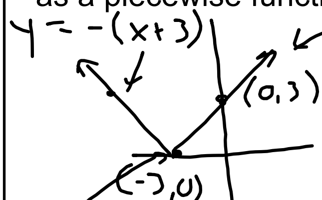
$= \frac{1}{12}$

Sum of Cubes:
$(A + B)(A^2-AB+B^2)$
Difference of Cubes:
$(A - B)(A^2 + AB + B^2)$

This skill will be VERY important in future courses when you are asked to complete related rate problems. It will also be useful here when we begin to do derivatives.

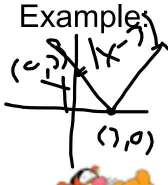
Option 4: Use One-Sided Limits

When you are asked to find the limit of an absolute value function you need to think about the two cases for that function (to the left and right of the cusp of the graph). What does the graph of f(x) = |x + 3| look like? How can I write this as a piecewise function instead?



$f(x)=\begin{cases} x+3, & \text{if } x>-3 \\ -(x+3), & \text{if } x<-3 \end{cases}$

Example: Use one-sided limits to determine if $\lim_{x\to 3} \frac{|x-3|}{x-3}$ exists.

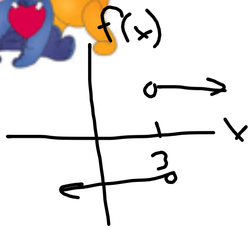


$f(x)=\begin{cases} \frac{x-3}{x-3}, & \text{if } x>3 \\ \frac{-(x-3)}{x-3}, & \text{if } x<3 \end{cases}$

$=\begin{cases} 1, & \text{if } x>3 \\ -1, & \text{if } x<3 \end{cases}$

$\lim_{x\to 3^-} f(x) = -1$

$\lim_{x\to 3^+} f(x) = 1$



limit at 3
Does not exist
(jump)