p.18 #22

$$y = \frac{1}{3}x^3 - 5x - \frac{4}{x} \qquad m = 0$$

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3}(x+h)^3 - 5(x+h) - \frac{4}{x+h} - \left(\frac{1}{3}x^3 - 5x - \frac{4}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3}\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - 5x - 5h - \frac{4}{x+h} - \frac{1}{3}x^3 + 5x + \frac{4}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{\frac{1}{3}x^3} + x^2h + xh^2 + \frac{1}{3}h^3 - \cancel{5x} - 5h - \frac{4}{x+h} - \cancel{\frac{1}{3}x^3} + \cancel{5x} + \frac{4}{x}}{h}$$

$$= \lim_{h\to 0} \frac{x^2h + xh^2 + \frac{1}{3}h^3 - 5h - \frac{4}{x+h} + \frac{4}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3x^3h(x+h) + 3x^2h^2(x+h) + xh^3(x+h) - 15xh(x+h) - 12x + 12(x+h)}{3x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3x^4h + 3x^3h^2 + 3x^3h^2 + 3x^2h^3 + x^2h^3 + xh^4 - 15x^2h - 15xh^2 - \cancel{12x} + \cancel{12x} + 12h}{3x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{3x^4h + 6x^3h^2 + 4x^2h^3 + xh^4 - 15x^2h - 15xh^2 + 12h}{3x(x+h)} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{3x^4 + \cancel{6x^3h} + \cancel{4x^2h^2} + \cancel{xh^3} - 15x^2 - 15xh + 12}{3x(x+h)}$$

Let $h = 0$

$$= \frac{3x^4 - 15x^2 + 12}{3x^2} \Bigg\} \text{Tangent slope}$$

$m = 0$

$$\frac{3x^4 - 15x^2 + 12}{3x^2} = 0$$

$$x^2\left(\frac{x^4 - 5x^2 + 4}{x^2}\right) = (0)x^2$$

$$x^4 - 5x^2 + 4 = 0 \qquad x^2 - 5x + 4 = (x-4)(x-1)$$

$$(x^2 - 4)(x^2 - 1) = 0$$

$$(x+2)(x-2)(x+1)(x-1) = 0$$

Turning points :)

Wednesday, February 5, 2020

1.3 Rates of Change

Rates of change have many applications in mathematics, science, and business. We are often concerned with the way that two variables change with respect to one another, so there are many practical applications of average (secant slope) and instantaneous (tangent slope) rates of change.

Velocity as a Rate of Change

Average velocity refers to the change in distance with respect to the change in time, or $\Delta s/\Delta t$.

Instantaneous velocity (usually referred to as velocity) is the rate of change in distance at a specific point in time, or $\lim\limits_{h \to 0}(\Delta s/\Delta t)$.

Example 1: A toy rocket is launched straight up so that its height, s, in meters, at time, t, in seconds, can be modelled by $s(t) = -5t^2 + 30t + 2$. What is the velocity of the rocket at t = 3?

$$s'(3) = \lim_{h \to 0} \frac{\overbrace{-5(3+h)^2}^{-45}\overbrace{+30(3+h)}^{90}+2-(-5t^2+30t+2)}{h}$$

limit → tangent slope

$$= \lim_{h \to 0} \frac{-5(9+6h+h^2)+90+30h+2-47}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-45}-\cancel{30h}-5h^2+\cancel{30h}+\cancel{45}}{h}$$

$$= \lim_{h \to 0} \frac{-5h^2}{h}$$

$$= \lim_{h \to 0}(-5h)$$

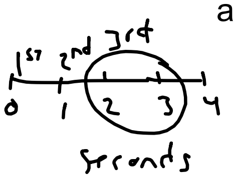
→ Let $h = 0$

$s'(3) = 0$

(vertex)

→ secant slope!

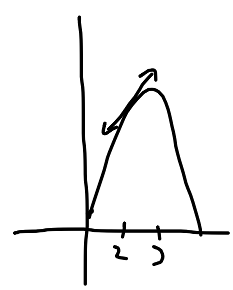
Example 2: For the same rocket, determine the average velocity in the third second of its flight. How are these questions different from one another?



$$m = \frac{s(3)-s(2)}{3-2}$$

$$= \frac{47-42}{1}$$

$$= 5\,m/s$$



This question has an interval so we can find slope between two point.

Other Applications of Rates of Change

Velocity is not the only application of rate of change, even though it will likely feel most familiar to those of you that have taken physics. We can apply the same principles to any relationship that can be modelled by a function containing two interdependent variables.

Example 3: The temperature, T, in degrees Celsius, varies with the height, in kilometers, above the Earth's surface according to the equation $T(h) = \frac{60}{h+2}$. Find the rate of change of the temperature with respect to height at a height of 6 km.

$$T'(6) = \lim_{h\to 0} \frac{T(6+h) - T(6)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{60}{h+8} - \frac{15}{2}}{h}$$
$$= \lim_{h\to 0} \frac{120 - 15(h+8)}{2(h+8)} \div h$$
$$= \lim_{h\to 0} \frac{120 - 15h - 120}{2(h+8)} \times \frac{1}{h}$$

$$T'(6) = \lim_{h\to 0} \frac{-15}{2h+16}$$

Let $h = 0$

$$T'(6) = \frac{-15}{16}\ {}^{\circ}C/km$$

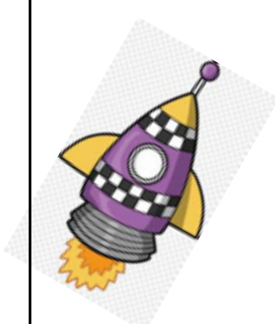
An Alternative Way to Find Instantaneous Rate of Change

Rather than using the difference quotient, we can also consider an alternative process for finding the tangent slope. To do this, we leave 'x' in the function, rather than plugging in (a + h), and then we consider the limit of the function as x approaches 'a'.

$$\boxed{\lim_{x\to a} \frac{f(x) - f(a)}{x - a}}$$

We are going to try this using the first example:

Example 1: A toy rocket is launched straight up so that its height, s, in meters, at time, t, in seconds, can be modelled by $s(t) = -5t^2 + 30t + 2$. What is the velocity of the rocket at t = 3?



$$s'(t) = \lim_{t\to 3} \frac{s(t) - s(3)}{t-3}$$
$$= \lim_{t\to 3} \frac{-5t^2 + 30t + 2 - 47}{t-3}$$
$$= \lim_{t\to 3} \left(\frac{-5t^2 + 30t - 45}{t-3} \right)$$
$$= \lim_{t\to 3} \frac{-5(t^2 - 6t + 9)}{t-3}$$
$$= \lim_{t\to 3} \frac{-5(t-3)^{\cancel{2}}}{\cancel{t-3}}$$
$$= \lim_{t\to 3} -5(t-3)$$

Let $t = 3$

$$s'(3) = 0$$