Friday, February 28, 2020

2.5 The Derivatives of Composite Functions


1) Advanced Functions Review/Preview

A **composite function** is basically a function within a function.

Notation: $(f \circ g)(x) = f[g(x)]$ → g(x) is the inner function, f(x) is the outer function

Reads: f of g at x is equal to f at g at x.

For example, if $f(x) = 3x^2 + 1$ and $g(x) = 2x - 3$, then:

$(f \circ g)(x) = 3(2x + 3)^2 + 1$ and $(g \circ f)(x) = 2(3x^2 + 1) - 3$

The process of creating composite functions is called composition. You have been working with composite functions already, we just may not have explicitly told you that. For example, given the composite function $h(x) = 2(x - 4)^3$, identify f(x) and g(x).

$f(x) = 2x^3 \quad g(x) = x - 4$

Advanced Functions Review:

Given that $f(x) = \sqrt{x}$ and $g(x) = x - 3$, find each of the following:

a) $f[g(4)]$
$= \sqrt{4-3}$
$= 1$

b) $g[f(4)]$
$= \sqrt{4} - 3$
$= -1$

c) $f[g(x)]$
$= \sqrt{x-3}$

d) $g[f(x)]$
$= \sqrt{x} - 3$

Is composition a commutative process?
↳ order does not matter

No → the order of the functions matters!

2) Finding Derivatives for Composite Functions

We can find derivatives for composite functions using the chain rule. We have already seen this with the 'power of a function rule.

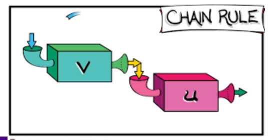
What do you think that you should do if you want to find the derivative of

$h(x) = (2x^2 + x)^{\frac{2}{3}}$ ?

$$h'(x) = \frac{2}{3}(2x^2+x)^{-1/3}\cdot(4x+1)$$

derivative of inside

derivative of outside




The Chain Rule

If f and g are functions that have derivatives, then the composite function h(x) = f[g(x)] has a derivative given by:

$$h'(x) = f'[g(x)]g'(x).$$

In words: The derivative of the outer function in terms of the inner function times the derivative of the inner function.

In Leibniz notation: $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Proof: Use the definition of the derivative to differentiate h(x) = f[g(x)].

$$h'(x) = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}$$

$$g'(x) = \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}\cdot\frac{g(x+h)-g(x)}{g(x+h)-g(x)}$$

Multiply by one.

Assume $g(x+h)-g(x)\neq 0$

$$= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}$$

As $h\to 0$, $g(x+h)-g(x)\to 0$ ???

$$= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

prop. of limits

Let $k = g(x+h) - g(x)$

$g(x+h) = k + g(x)$

$$= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}\cdot g'(x)$$

$$= \lim_{k\to 0}\frac{f(g(x)+k) - f(g(x))}{k}\cdot g'(x)$$

$$= f'(g(x))\cdot g'(x)$$

Practice Using the Chain Rule (and all of the other rules...)

1) Differentiate each of the following.

a) $h(x) = \sqrt{x^3 + x}$

$$h(x) = (x^3 + x)^{1/2}$$

$$h'(x) = \frac{1}{2}(x^3 + x)^{-1/2}(3x^2 + 1)$$

$$= \frac{3x^2 + 1}{2(x^3 + x)^{1/2}}$$

Quotient Rule:

b) $h(x) = \left(\frac{1 + x^2}{1 - x^2}\right)^8$

$$h'(x) = 8\left(\frac{1 + x^2}{1 - x^2}\right)^7 \left[\frac{2x(1 - x^2) - (-2x)(1 + x^2)}{(1 - x^2)^2}\right]$$

$$= 8\left(\frac{1 + x^2}{1 - x^2}\right)^7 \left(\frac{4x}{(1 - x^2)^2}\right)$$

$$= \frac{32x(1 + x^2)^7}{(1 - x^2)^9}$$

2) If $y = u^3 - 2u + 1$, where $u = 2\sqrt{x}$, find $^{dy}/_{dx}$ at $x = 4$. (Use Leibniz notation!)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{du} = 3u^2 - 2$$

$$u = 2x^{1/2}$$

$$\frac{du}{dx} = x^{-1/2}$$

When $x = 4$

$u = 2\sqrt{4}$

$u = 4$

$$\frac{dy}{dx} = (3u^2 - 2)(x^{-1/2})$$

$$\frac{dy}{dx} = (3(16) - 2)\left(\frac{1}{\sqrt{4}}\right)$$

$$= 23$$

3) The function $s(t) = (t^3 + t^2)^{\frac{1}{2}}$ represents the displacement, s, in meters, of a particle moving along a straight line after t seconds. Determine the velocity of the particle at t and when t = 2.

$$s'(t) = \frac{1}{2}(t^3 + t^2)^{-1/2} \cdot (3t^2 + 2t)$$

$$v(t) = s'(t) = \frac{3t^2 + 2t}{2(t^3 + t^2)^{1/2}}$$

$$v(2) = \frac{3(2)^2 + 2(2)}{2(8 + 4)^{1/2}}$$

$$= \frac{16}{2\sqrt{12}}$$

$$= \frac{8}{\sqrt{12}}$$

$$= \frac{8}{2\sqrt{3}}$$

$$= \frac{4}{\sqrt{3}} \text{ or } \frac{4\sqrt{3}}{3} \text{ m/s}$$

**CHAIN RULE:**

IF $h(x) = g(f(x))$

$h'(x) = g'(f(x)) \cdot f'(x)$