Monday, February 24, 2020

2.3 The Product Rule

Expanding and simplifying is annoying sometimes, and also increases the chances of a sign error, etc. The product rule will allow us to find the derivative of the product of two functions without expanding, and will also allow us to find the derivative of the product of two functions that we cannot apply the power rule to after we expand and simplify (once we know how to differentiate other types of functions).

Examples: $f(x) = (x - 5)^5(2x - 3)^3$ $\qquad$ $g(x) = (2^x - 3)(x - \sin x)$

The Product Rule

In words: The derivative of the product of two functions is equal to the derivative of the first function times the second function plus the first function times the derivative of the second function.

Algebraic representation: If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$

Try it! Find the derivative of $p(x) = (x^2 - 5x + 1)(2x^3 - x^2 + x)$

$f(x) = x^2 - 5x + 1$

$f'(x) = 2x - 5$

$g(x) = 2x^3 - x^2 + x$

$g'(x) = 6x^2 - 2x + 1$

$p'(x) = (2x - 5)(2x^3 - x^2 + x) + (6x^2 - 2x + 1)(x^2 - 5x + 1)$

$p'(x) = 4x^4 - 2x^3 + 2x^2 - 10x^3 + 5x^2 - 5x + 6x^4 - 30x^3 + 6x^2 - 2x^3 + 10x^2 - 2x + x^2 - 5x + 1$

$= 10x^4 - 44x^3 + 24x^2 - 12x + 1$

$p'(x) = f'(x)g(x) + g'(x)f(x)$

Spoiler Alert:

We will introduce a "fancy zero" here! $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

We need this.

Proof of the Product Rule:

Differentiate p(x) = f(x)g(x) from first principles.

$$p'(x) = \lim_{h\to 0}\left[\frac{f(x+h)\cdot g(x+h) - f(x)\,g(x)}{h}\right]$$

We need to subtract/add $f(x)\cdot g(x+h)$

$$p'(x) = \lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)g(x)}{h}\right]$$

$$p'(x) = \lim_{h\to 0}\left[\frac{g(x+h)\big(f(x+h)-f(x)\big) + f(x)\big(g(x+h)-g(x)\big)}{h}\right]$$

$$p'(x) = \lim_{h\to 0}\frac{g(x+h)\big(f(x+h)-f(x)\big)}{h} + \lim_{h\to 0}\frac{f(x)\big(g(x+h)-g(x)\big)}{h}$$

$$p'(x) = \lim_{h\to 0} g(x+h)\cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\cdot \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$p'(x) = g(x)f'(x) + f(x)g'(x)$$

Example:

Find the value of p'(-1) for the function p(x) = $(5x^3 + 7x^2 + 3)(2x^2 + x + 6)$.

$$p'(x) = (15x^2 + 14x)(2x^2 + x + 6) + (4x+1)(5x^3 + 7x^2 + 3)$$

$$p'(-1) = (15-14)(2-1+6) + (-3)(-5+7+3)$$

$$= 7 - 15$$

$$= -8$$

Extending the Product Rule

1) More than Two Functions

1st

2nd (use the product rule to differentiate)

Find an expression for p'(x) if p(x) = f(x)[g(x)h(x)].

$$p'(x) = f'(x)g(x)h(x) + f(x)\left[g'(x)h(x) + g(x)h'(x)\right]$$

$$= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

2) <u>A Whole Function is Raised to a Power</u>

On the first page, I showed the example $f(x) = (x - 5)^5(2x - 3)^3$. This is the product of two linear functions that are each raised to a power (or it's the product of a quintic polynomial and a cubic polynomial, but that would take forever to expand and simplify!).

Instead of expanding, we can use the **Power of a Function Rule**, which states that the derivative of the power of a function is equal to the derivative of the power function times the derivative of the inner function.

$$\text{If } f(x) = [g(x)]^n \text{ then } f'(x) = n[g(x)]^{n-1}g'(x).$$

For example, the derivative of $(2x - 3)^3$ would be $3(2x - 3)^2(2)$, which simplifies to $6(2x - 3)^2$.

Try the other half of the function given above, $(x - 5)^5$.

$$f'(x) = 5(x-5)^4(1)$$
$$= 5(x-5)^4$$

Now we can find the derivative of $f(x) = (x - 5)^5(2x - 3)^3$ using the product rule.

$$f'(x) = 5(x-5)^4(2x-3)^3 + 6(2x-3)^2(x-5)^5$$

Common factor!

$$= (x-5)^4(2x-3)^2\left(5(2x-3) + 6(x-5)\right)$$
$$= (x-5)^4(2x-3)^2(16x-45)$$

How do you feel about the resulting expression...?

Ugly... don't expand the binomials, but do common factor.

Often you will be asked to apply the power of a function rule, but will also be explicitly told NOT to simplify the resulting expression. Please follow the instructions!!

Example: Determine the equation of the tangent to the curve
$y = (x^3 - 5x + 2)(3x^2 - 2x)$ at the point $(-1, 30)$.

find y' when $x = -1$

$$y' = (3x^2 - 5)(3x^2 - 2x) + (6x - 2)(x^3 - 5x + 2)$$

When $x = -1$,

$$y' = (-2)(5) + (-8)(6)$$
$$y' = -58$$

$$y = mx + b$$
$$30 = -58(-1) + b$$
$$30 = 58 + b$$
$$-28 = b$$

$$\boxed{y = -58x - 28}$$