Thursday, February 20, 2020

2.2 The Derivatives of Polynomial Functions

Bellwork:

Find the derivative of each of the following functions from first principles. What do you notice?

1) $f(x) = 2x^2 + x$

$$f'(x) = \lim_{h\to 0} \frac{\left[2(x+h)^2 + (x+h) - (2x^2+x)\right]}{h}$$

$$= \lim_{h\to 0} \frac{(2x^2 + 4xh + 2h^2 + x + h - 2x^2 - x)}{h}$$

$$= \lim_{h\to 0} \frac{4xh + 2h^2 + h}{h}$$

$$= \lim_{h\to 0} 4x + 2h + 1$$

$$f'(x) = 4x + 1$$

2) $g(x) = -x^3 + x$

$$g'(x) = \lim_{h\to 0} \frac{\left[-(x+h)^3 + (x+h) - (-x^3+x)\right]}{h}$$

$$= \lim_{h\to 0} \frac{(-x^3 - 3x^2h - 3xh^2 - h^3 + x + h + x^3 - x)}{h}$$

$$= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3 + h}{h}$$

$$= \lim_{h\to 0} -3x^2 - 3xh - h^2 + 1$$

$$= -3x^2 + 1$$

Examine the following functions and their derivatives. Does it support what you noticed?

$f(x) = 4x^3 - 3x^2 + 5$; $f'(x) = 12x^2 - 6x$

$g(x) = 2x^4 + x^3 + 5x^2 - 2x$; $g'(x) = 8x^3 + 3x^2 + 10x - 2$

$h(x) = 4x^{-2} + x^{-1} + 3x^2$; $h'(x) = -8x^{-3} - x^{-2} + 6x$

We do not always (or ever really...) need to calculate derivatives from first principles. Today we will start looking at derivative rules that apply to polynomial functions. A power function is a function that can be written in the form $f(x) = x^n$. Keep in mind that you can rewrite radical and rational expressions as powers!

Example: Write $\sqrt{x^3}$ and $1/x^4$ as power functions.

$$f(x) = \sqrt{x^3} = (x^3)^{1/2} = x^{3/2}$$

$$f'(x) = \frac{3}{2}x^{1/2}$$

$$g(x) = \frac{1}{x^4} = x^{-4}$$

$$g'(x) = -4x^{-5} = -\frac{4}{x^5}$$

Derivative Rules

1) The **constant function rule** states that the derivative of a constant is zero.

$f(x) = c,\ f'(x) = 0$ $\qquad$ $f(x) = 6,\ f'(x) = 0$

2) The **power rule** states that for a power function, $f(x) = x^n$, $f'(x) = nx^{n-1}$. Basically, we bring the exponent in front and then subtract one.

3) The **constant multiple rule** states that if $f(x) = kg(x)$ where k is a constant, then $f'(x) = kg'(x)$.

Proof of the Constant Multiple Rule:

uses limits!

Let $f(x) = kg(x)$, and differentiate from first principles.

$$f'(x) = \lim_{h\to 0} \frac{kg(x+h) - kg(x)}{h}$$

$$= \lim_{h\to 0} \frac{k[g(x+h) - g(x)]}{h}$$ Common factor.

properties of limits

$$= \lim_{h\to 0} k \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$ Definition of the derivative.

$$= kg'(x)$$:)

4) The **sum rule** states that if functions p(x) and q(x) are differentiable, and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.

We need this.

5) The **difference rule** states that if functions p(x) and q(x) are differentiable, and $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.

$$p'(x) = \lim_{h\to 0} \frac{p(x+h) - p(x)}{h}$$

$$q'(x) = \lim_{h\to 0} \frac{q(x+h) - q(x)}{h}$$

Proof of the Difference Rule:

Let $f(x) = p(x) - q(x)$ and differentiate from first principles.

$$f'(x) = \lim_{h\to 0} \frac{p(x+h) - q(x+h) - (p(x) - q(x))}{h}$$

Group p's + q's

$$= \lim_{h\to 0} \frac{p(x+h) - p(x) - q(x+h) + q(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{p(x+h) - p(x)}{h} - \frac{q(x+h) - q(x)}{h} \right)$$

Property of Limits

$$= \lim_{h\to 0} \frac{p(x+h)-p(x)}{h} - \lim_{h\to 0} \frac{q(x+h)-q(x)}{h}$$

$$f'(x) = p'(x) - q'(x)$$:)

Practice Problems:

1) Differentiate the following functions.

a) $f(x) = 5x^2 - 9x^3$

$f'(x) = 10x - 27x^2$

b) $y = \frac{1}{x^3} - \frac{5}{x} + x^{-2}$

$y = x^{-3} - 5x^{-1} + x^{-2}$

$y' = -3x^{-4} + 5x^{-2} - 2x^{-3}$

c) $h(x) = \sqrt{2x} + \sqrt[3]{x} - 3x^{\frac{2}{5}}$

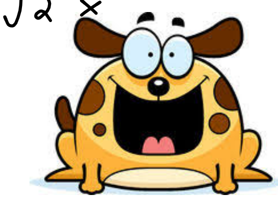
$= \sqrt{2}\,x^{1/2} + x^{1/3} - 3x^{2/5}$

$h'(x) = \frac{\sqrt{2}}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} - \frac{6}{5}x^{-3/5}$

$\sqrt{2x} = (2x)^{1/2}$

$= 2^{1/2}x^{1/2}$

$= \sqrt{2}\,x^{1/2}$



2) Determine the equation of the tangent to $y = -x^2 + 3x - 5$ at $x = -3$.

$y' = -2x + 3$

When $x = -3$

$y' = -2(-3) + 3$

$= 9$

$y = -(-3)^2 + 3(-3) - 5$

$= -23$

Pt. of tangency

$(-3, -23)$

$y = mx + b$

$-23 = 9(-3) + b$

$4 = b$

$y = 9x + 4$

3) Determine the point(s) on the graph $f(x) = -x^3 + 3x^2 - 2$ where the tangent(s) are horizontal. (Think about what we know about polynomial functions!)

(Turning points, $f'(x) = 0$)

$f'(x) = -3x^2 + 6x$

$-3x^2 + 6x = 0$

$-3x(x-2) = 0$

$x = 0 \quad x = 2$

$\therefore$ The tangents are horizontal when $x = 0$ and when $x = 2$.