Thursday, February 20, 2020

2.2 The Derivatives of Polynomial Functions

Bellwork:

Find the derivative of each of the following functions from first principles. What do you notice?

1) $f(x) = 2x^2 + x$

$$f'(x) = \lim_{h\to 0} \frac{[2(x+h)^2 + (x+h)] - (2x^2 + x)}{h}$$
$$= \lim_{h\to 0} \frac{(2(x^2 + 2xh + h^2) + x + h) - 2x^2 - x}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h - \cancel{2x^2} - \cancel{x}}{h}$$
$$= \lim_{h\to 0} 4x + 2h + 1$$
$$= 4x + 1$$

2) $g(x) = -x^3 + x$

$$g'(x) = \lim_{h\to 0} \frac{[-(x+h)^3 + (x+h) - (-x^3 + x)]}{h}$$
$$= \lim_{h\to 0} \frac{[-(x^3 + 3x^2h + 3xh^2 + h^3) + x + h + x^3 - x]}{h}$$
$$= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3 + h}{h}$$
$$= -3x^2 + 1$$

Examine the following functions and their derivatives. Does it support what you noticed?

$f(x) = 4x^3 - 3x^2 + 5$; $f'(x) = 12x^2 - 6x$

$g(x) = 2x^4 + x^3 + 5x^2 - 2x$; $g'(x) = 8x^3 + 3x^2 + 10x - 2$

$h(x) = 4x^{-2} + x^{-1} + 3x^2$; $h'(x) = -8x^{-3} - x^{-2} + 6x$

We do not always (or ever really...) need to calculate derivatives from first principles. Today we will start looking at derivative rules that apply to polynomial functions. A power function is a function that can be written in the form $f(x) = x^n$. Keep in mind that you can rewrite radical and rational expressions as powers!

Example: Write $\sqrt{x^3}$ and $1/x^4$ as power functions.

$f(x) = \sqrt{x^3}$

$f(x) = x^{3/2}$

$f'(x) = \frac{3}{2}x^{1/2}$

$g(x) = \frac{1}{x^4}$

$= x^{-4}$

$g'(x) = -4x^{-5}$

$= -\frac{4}{x^5}$

Derivative Rules

1) The **constant function rule** states that the derivative of a constant is zero.

$f(x) = c, f'(x) = 0$ $\qquad$ $f(x) = 6, f'(x) = 0$

2) The **power rule** states that for a power function, $f(x) = x^n$, $f'(x) = nx^{n-1}$. Basically, we bring the exponent in front and then subtract one.

3) The **constant multiple rule** states that if $f(x) = kg(x)$ where k is a constant, then $f'(x) = kg'(x)$.

Proof of the Constant Multiple Rule: (Prove from first principles means use limit!)

Let $f(x) = kg(x)$, and differentiate from first principles.

$$f'(x) = \lim_{h\to 0} \frac{kg(x+h) - kg(x)}{h}$$

Prop. of limits

$$= k\boxed{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}$$

← definition of the derivative → all of that stuff just equals $g'(x)$ (by definition)

$$= kg'(x)$$

4) The **sum rule** states that if functions p(x) and q(x) are differentiable, and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.

5) The **difference rule** states that if functions p(x) and q(x) are differentiable, and $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.

$$p'(x) = \lim_{h\to 0} \frac{p(x+h) - p(x)}{h}$$

Proof of the Difference Rule:

Let $f(x) = p(x) - q(x)$ and differentiate from first principles.

$$f'(x) = \lim_{h\to 0} \left[\frac{p(x+h) - q(x+h) - (p(x) - q(x))}{h}\right]$$

$$= \lim_{h\to 0} \left[\frac{p(x+h) - p(x) - q(x+h) + q(x)}{h}\right]$$

$$= \lim_{h\to 0} \left[\frac{p(x+h) - p(x)}{h} - \frac{q(x+h) - q(x)}{h}\right]$$

Property of limits

$$= \lim_{h\to 0} \frac{p(x+h) - p(x)}{h} - \lim_{h\to 0} \frac{q(x+h) - q(x)}{h}$$

$$= p'(x) - q'(x)$$:)

Practice Problems:

1) Differentiate the following functions.

a) $f(x) = 5x^2 - 9x^3$

$f'(x) = 10x - 27x^2$

b) $y = \frac{1}{x^3} - \frac{5}{x} + x^{-2}$

$= x^{-3} - 5x^{-1} + x^{-2}$

$y' = -3x^{-4} + 5x^{-2} - 2x^{-3}$

$= -\frac{3}{x^4} + \frac{5}{x^2} - \frac{2}{x^3}$

c) $h(x) = \sqrt{2x} + \sqrt[3]{x} - 3x^{\frac{2}{5}}$

$= (2x)^{1/2} + x^{1/3} - 3x^{2/5}$

$= \sqrt{2}\, x^{1/2} + x^{1/3} - 3x^{2/5}$

$h'(x) = \frac{\sqrt{2}}{2} x^{-1/2} + \frac{1}{3} x^{-2/3} - \frac{6}{5} x^{-3/5}$

$= \frac{\sqrt{2}}{2x^{1/2}} + \frac{1}{3x^{2/3}} - \frac{6}{5x^{3/5}}$

2) Determine the equation of the tangent to $y = -x^2 + 3x - 5$ at $x = -3$.

$y' = -2x + 3 \leftarrow$ tangent slope

When $x = -3$:

$y' = -2(-3) + 3$

$y' = 9$

$y = -(-3)^2 + 3(-3) - 5$

$= -9 - 9 - 5$

$= -23$

$y = mx + b$

$-23 = 9(-3) + b$

$-23 = -27 + b$

$4 = b$

$y = 9x + 4$

3) Determine the point(s) on the graph $f(x) = -x^3 + 3x^2 - 2$ where the tangent(s) are horizontal. (Think about what we know about polynomial functions!)

$f'(x) = 0$

$f'(x) = -3x^2 + 6x$

$-3x^2 + 6x = 0$

$-3x(x - 2) = 0$

$x = 0, \; x = 2$

$\therefore$ The tangent slope is zero when $x = 0$ and when $x = 2$.