Wednesday, February 19, 2020

2.1 The Derivative Function

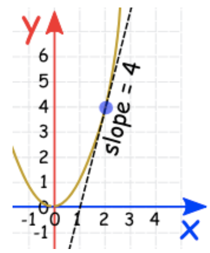
The derivative of a function is given by the limit of the difference quotient. It is also the instantaneous rate of change for a function with respect to x.

"f prime at x" → $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

OR

Liebniz notation

"y prime" → $$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$



We have already done the math that goes with this, and we have also talked about a lot of the theory. Recall that you can find the tangent slope at a specific point by subbing a value for x, a. Using this limit to find the instantaneous rate of change is called **determining the derivative from first principles**.

Example: Determine the derivative of f(x) = x^3 from first principles.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

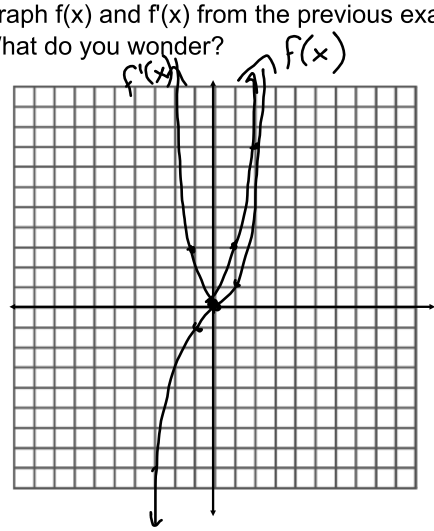
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

Let $h = 0$: $f'(x) = 3x^2$

Thinking of the Derivative as a Function

Graph f(x) and f'(x) from the previous example. What do you notice? What do you wonder?



$f(x) \to$ cubic

$f'(x) \to$ quadratic

Decreased the degree by 1.

Example: Find the derivative of f(x) = √x from first principals. Then find the equation of the tangent line to the graph at the point (4, 2).

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Let h=0

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Slope: $f'(4) = \frac{1}{2(2)}$

$= \frac{1}{4}$

$y = mx + b$

$2 = \left(\frac{1}{4}\right)(4) + b$

$2 = 1 + b$

$1 = b$

$$\boxed{y = \frac{1}{4}x + 1}$$

What do you think we mean if we ask you to find the equation for the normal to the graph at a given value of x? Find an equation for the normal to our tangent line from the previous example.

⊥ to the line.

$m_\perp = -4$

Pt: (4, 2)

$2 = -4(4) + b$

$2 = -16 + b$

$18 = b$

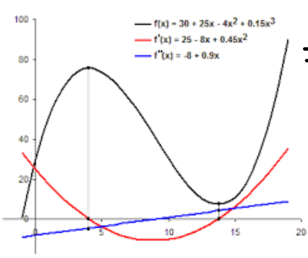
$y = -4x + 18$

Example: Find the derivative of $f(x) = \dfrac{2x+1}{x-3}$ from first principles.

$$f'(x) = \lim_{h \to 0} \frac{\frac{2(x+h)+1}{x+h-3} - \frac{2x+1}{x-3}}{h}$$

$$= \lim_{h \to 0} \frac{(2x+2h+1)(x-3) - (2x+1)(x+h-3)}{(x+h-3)(x-3)} \times \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} - \cancel{6x} + \cancel{2xh} - 6h + \cancel{x} - \cancel{3} - (\cancel{2x^2} + \cancel{2xh} - \cancel{6x} + \cancel{x} + h - \cancel{3})}{(x+h-3)(x-3)}$$



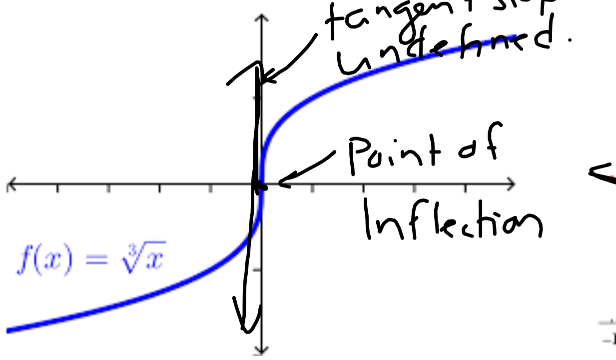
$$= \lim_{h \to 0} \frac{-7\cancel{h}}{(x+h-3)(x-3)} \times \frac{1}{\cancel{h}}$$

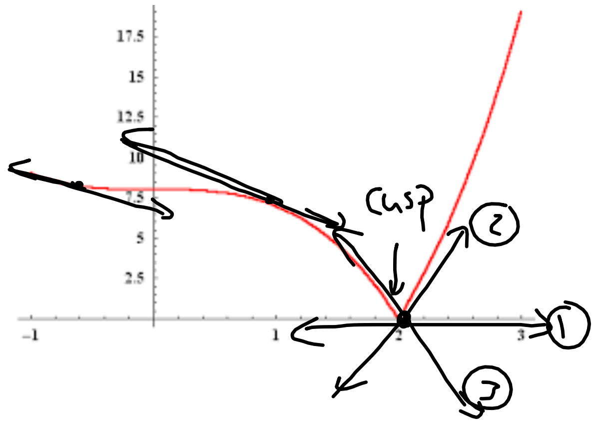
$$= \lim_{h \to 0} \frac{-7}{(x+h-3)(x-3)}$$

Let h=0 $f'(x) = \dfrac{-7}{(x-3)^2}$

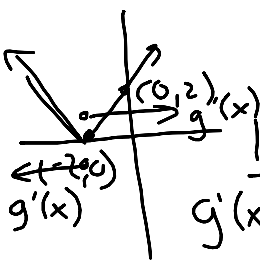
The Existence of Derivatives

A function, f, is differentiable at a if f'(a) exists. When there is a point where the function is not differentiable, then the derivative does not exist (much like limits). This occurs at jump discontinuities, as we already saw. It also occurs when the tangent is vertical (slope is undefined) and at a cusp (we'll talk about this in detail using the picture below!).





Example: Show, from first principles, that the function g(x) = |x+2| is not differentiable at x = -2.



$$g(x) = \begin{cases} x+2, & \text{if } x > -2 \\ -x-2, & \text{if } x < -2 \end{cases}$$

If $x > -2$

$$g'(x) = \lim_{h \to 0} \frac{x+h+2-(x+2)}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

If $x < -2$

$$g'(x) = \lim_{h \to 0} \frac{-(x+h)-2-(-x-2)}{h} = \lim_{h \to 0} -\frac{h}{h} = \lim_{h \to 0} -1 = -1$$

If a function is not differentiable, does it mean that it is not continuous?

No, a function can be continuous but not differentiable.