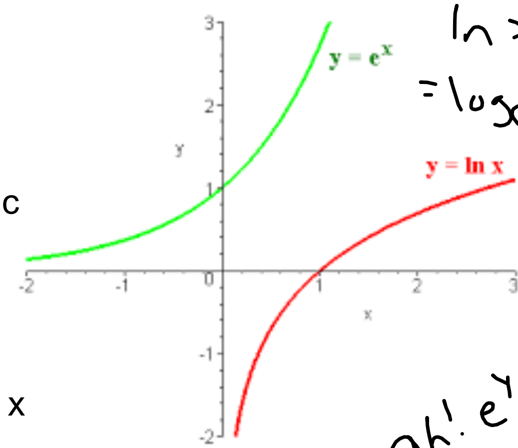
Date: ____________________

Calculus Appendix 3: The Natural Logarithm and its Derivative

The natural logarithm function, $y = \ln x$, is just the inverse of the exponential function $y = e^x$.

$\ln x = \log_e x$



Recall that you can write a logarithmic function in exponential form:

$y = \log_b x \longrightarrow b^y = x$

This means that we can rewrite $y = \ln x$ as $x = e^y$. (The base of the natural logarithm is 'e')

Oh! $e^y = x$

We can apply implicit differentiation to determine the derivative of this expression.

Differentiate $x = e^y$

$$1 = e^y \cdot \frac{dy}{dx}$$

$$\frac{1}{e^y} = \frac{dy}{dx}$$

$$\frac{1}{x} = \frac{dy}{dx}$$

You can now add the derivative of $y = \ln x$ to your derivative tool box. Please remember to apply the derivative rules that you already know when working with composite functions that include the natural logarithm!!

Practice Problems:

1) Find the derivative for each of the following functions:

a) $y = \ln 2x$

$$y' = \frac{1}{2x} \cdot 2$$
$$= \frac{1}{x}$$

b) $y = x^3 \ln x$

$$y' = 3x^2 \ln x + x^3\left(\frac{1}{x}\right)$$
$$= 3x^2 \ln x + x^2$$
$$= x^2(3\ln x + 1)$$

c) $y = \ln (x^2-3x)$

$$y' = \frac{1}{x^2-3x}(2x-3)$$
$$= \frac{2x-3}{x^2-3x}$$

2) Determine the equation of a tangent to the curve $y = x^2 \ln x$ at the point (1, 0).

$e^{?} = 1$

$$\frac{dy}{dx} = 2x\ln x + x^2\left(\frac{1}{x}\right)$$

$$= 2x\ln x + x$$

$$\frac{dy}{dx} = \overbrace{2(1)\ln 1}^{0} + 1$$

$$= 1$$

$$y = mx + b$$

$$0 = (1)(1) + b$$

$$-1 = b$$

$$\boxed{y = x - 1}$$

Revisiting Derivatives of Exponential Functions

We saw in chapter 5 that the derivative of $y = a^x$ is $y' = a^x \ln a$. Now that we know how to use implicit differentiation and how to take the derivative of the natural logarithm we can see why this is the case.

Example: Differentiate $y = 2^x$ by writing it in terms of the natural logarithm.

Take natural log of both sides: $\ln y = \ln 2^x$

But $y = 2^x$!!

Apply log laws: $\ln y = x \ln 2$

Implicit Diff wrt x: $\frac{1}{y} \cdot \frac{dy}{dx} = \ln 2$

Isolate $\frac{dy}{dx}$: $\frac{dy}{dx} = y \ln 2$

$$\frac{dy}{dx} = 2^x \ln 2$$