4.1 Increasing and Decreasing Functions

1a) $f(x) = x^3 + 6x^2 + 1$

$f'(x) = 3x^2 + 12x$

$3x(x+4) = 0$

$x = 0 \quad x = -4$

$f(0) = 1 \quad f(-4) = 33$

d) $f(x) = \frac{5x}{x^2+1}$

$f'(x) = \frac{5(x^2+1) - 5x(2x)}{(x^2+1)^2}$

$0 = \frac{5x^2+5-10x^2}{(x^2+1)^2}$

$-5x^2 + 5 = 0$

$x^2 = 1$

$x = \pm 1$

$f(1) = 5/2 \quad f(-1) = -5/2$

4e) $f(x) = 3x^4 + 4x^3 - 12x^2$

$f'(x) = 12x^3 + 12x^2 - 24x$

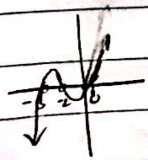
$12x(x^2 + x - 2) = 0$

$12x(x+2)(x-1) = 0$

TP @ $x = 0, -2, 1$

↑ $x \in (-2, 0), (1, \infty)$

↓ $x \in (-\infty, -2), (0, 1)$



5. $f'(x) = (x-1)(x+2)(x+3)$

↳ zeros @ $x = 1, -2, -3$ + cubic w/ ⊕LC

$\therefore$ ↑ $x \in (-3, -2), (1, \infty)$

↓ $x \in (-\infty, -3), (-2, 1)$

b) $f(x) = \sqrt{x^2+4}$

$f'(x) = \frac{1}{2}(x^2+4)^{-1/2}(2x)$

$0 = \frac{x}{\sqrt{x^2+4}}$

$x = 0 \quad f(0) = 2$

2. ↑ when $f'(x)$ is ⊕,

↓ when $f'(x)$ is ⊖.

3a) ↑ $x \in (-\infty, -1), (2, \infty)$

↓ $x \in (-1, 2)$

b) ↑ $x \in (-1, 1)$

↓ $x \in (-\infty, -1), (1, \infty)$

c) ↑ $x \in (-\infty, -2),$

↓ $x \in (-2, 2), (2, \infty)$

d) ↓ $x \in (-\infty, -1), (2, 3)$

↑ $x \in (-1, 2), (3, \infty)$

f) $f(x) = x^4 + x^2 - 1$

$f'(x) = 4x^3 + 2x$

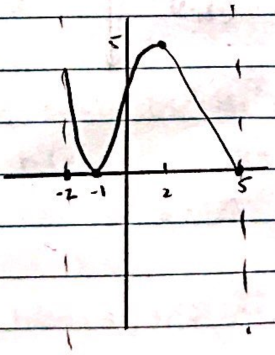
$2x(x^2+1) = 0$

TP @ $x = 0$

↑ $x \in (0, \infty)$

↓ $x \in (-\infty, 0)$

6.



c) $f(x) = (2x-1)^2(x^2-9)$

$f'(x) = 2(2x-1)(2)(x^2-9) + (2x-1)^2(2x)$

$= 4(2x-1)(x^2-9) + 2x(2x-1)^2$

$2(2x-1)[2(x^2-9) + x(2x-1)] = 0$

$2(2x-1)(2x^2-18+2x^2-x) = 0$

$2(2x-1)(4x^2-x-18) = 0$

$2(2x-1)(4x-9)(x+2) = 0$

$x = \frac{1}{2} \quad x = 9/4 \quad x = -2$

$f(\frac{1}{2}) = 0 \quad f(\frac{9}{4}) = -\frac{63}{16} \quad f(-2) = -125$

4a) $f(x) = x^3 + 3x^2 + 1$ (LC ⊕, odd degree)

$f'(x) = 3x^2 + 6x$

$3x(x+2) = 0$ T.P @ $x = 0, x = -2$

↑ $x \in (-\infty, -2), (0, \infty)$

↓ $x \in (-2, 0)$

b) $f(x) = x^5 - 5x^4 + 100$ (LC ⊕, odd dg)

$f'(x) = 5x^4 - 20x^3$

$5x^3(x-4) = 0$ TP @ $x = 0, x = 4$

↑ $x \in (-\infty, 0), (4, \infty)$

↓ $x \in (0, 4)$

c) $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$ VA @ $x = 0$

$\frac{(x+1)(x-1)}{x^2} = 0$ TP @ $x = \pm 1$

↑ $x \in (-\infty, -1), (1, \infty)$

↓ $x \in (-1, 0), (0, 1)$

d) $f(x) = \frac{x-1}{x^2+3}$ (no VA)

$f'(x) = \frac{x^2+3-2x(x-1)}{(x^2+3)^2}$

$\frac{-x^2+2x+3}{(x^2+3)^2} = 0$

$-(x+1)(x-3) = 0$ TP @ $x = -1, x = 3$

↑ $x \in (1, 3)$

↓ $x \in (-\infty, -1), (3, \infty)$.

7. $f(x) = x^3 + ax^2 + bx + c$ TP @ (3, 18), (1, -14)

$f'(x) = 3x^2 + 2ax + b$

$3(-3)^2 + 2a(-3) + b = 0$ $3 + 2a + b = 0$

$-6a + b = -27$ $2a + b = -3$

$b = +6a - 27$ $2a + 6a - 27 = -3$

$b = 36 - 27$ $8a = 24$

$\boxed{b = -9}$ $\boxed{a = 3}$

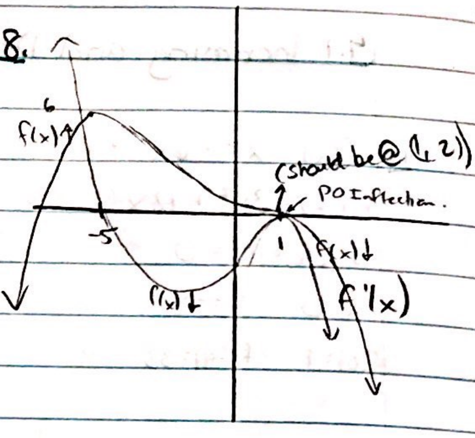
$f(x) = x^3 + 3x^2 - 9x + c$

$-14 = 1 + 3 - 9 + c$

$+9 = c$

$\boxed{f(x) = x^3 + 3x^2 - 9x - 9}$

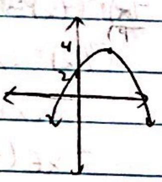
8.



9a i) $x \in (-\infty, 4)$

ii) $x \in (4, \infty)$

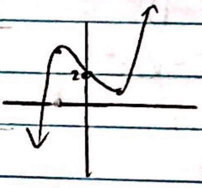
iii) $x = 4$



b i) ↑ $(-\infty, -1), (1, \infty)$

ii) ↓ $(-1, 1)$

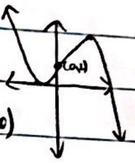
iii) $x = \pm 1$



c i) ↑ $(-2, 3)$

ii) ↓ $(-\infty, -2), (-3, \infty)$

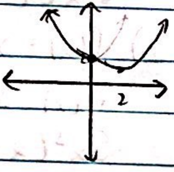
iii) $x = -2, x = 3$



d i) ↑ $(2, \infty)$

ii) ↓ $(-\infty, 2)$

iii) $x = 2$



10. $f(x) = ax^2 + bx + c$, $a > 0$

↑ opens up.

$f'(x) = 2ax + b$

$2ax + b = 0$

$x = -b/2a$ (turning point)

11. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32$

$4(x^3 - 8) = 0$

$\boxed{x = 2}$

$f(2) = 16 - 64 + 4$

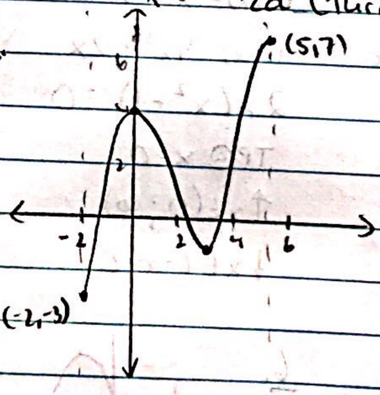
$= -44$

min @ (2, -44)

↓ $x \in (-\infty, 2)$

↑ $x \in (2, \infty)$

12.



4.2 Critical Points, Local Maxima, and Local Minima

1. Find the values where $f'(x)=0$ or $f'(x)$ is undefined.

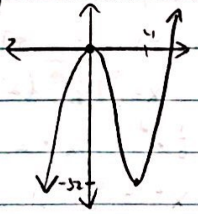
2a) Take the derivative, then set it equal to zero and solve.

b) $y' = 3x^2-12x$

$3x(x-4)=0$

$x=0,\ x=4$

$(0,0),\ (4,-32)$



3a) $y = x^4-8x^2$

$y' = 4x^3-16x$

$4x(x^2-4)=0$

$x=0,\ \pm 2$

$(0,0),\ (2,-16),\ (-2,-16)$

Let $x=-3, -1, 1, 3$ (sub into y')

$x=-3,\ y'=-60$ $x=-1,\ y'=12$

∴ $(-2,-16)$ is a local min

$x=1,\ y'=-12$

∴ $(0,0)$ is a local max.

$x=3,\ y'=60$

∴ $(2,16)$ is a local min

b) $f(x)=\frac{2x}{x^2+9}$

No V.A

$f'(x) = \frac{2(x^2+9)-2x(2x)}{(x^2+9)^2} = \frac{-2x^2+18}{(x^2+9)^2}$

$-2x^2+18=0$

$-2(x^2-9)=0$

$x=\pm 3$.

$f(3)=\frac{1}{3}$, $f(-3)=-\frac{1}{3}$

Check $f'(-4), f'(-2), f'(2), f'(4)$

$f'(-4) = -\frac{14}{625}$ $f(-2)=\frac{10}{169}$

∴ $(-3,-\frac{1}{3})$ is a local min

$f'(2)=\frac{10}{169}$ $f'(4) = -\frac{14}{625}$

∴ $(3,\frac{1}{3})$ is a local max.

3c) $y = x^3+3x^2+1$

$y' = 3x^2+6x$

$3x(x+2)=0$

$x=0,\ x=-2$

$(0,1),\ (-2,5)$

Let $x=-3, -1, 1$

$x=-3,\ y'=9$ $x=-1,\ y'=-3$

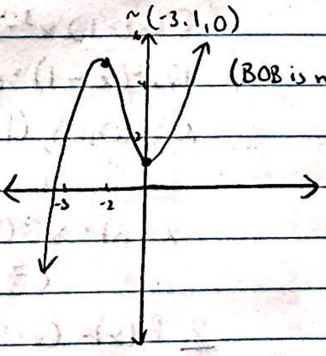
∴ $(2,5)$ is a local max

$x=1,\ y'=9$

∴ $(0,1)$ is a local min

4c) x-int: $x^3+3x^2+1=0$ y-int: $(0,1)$

~$(-3.1,0)$



(BOB is not right!)

5a) $h(x) = -6x^3+18x^2+3$

$h'(x) = -18x^2+36x$

$-18x(x-2)=0$

$x=0,\ x=2$

$h(0)=3$, $h(2)=27$

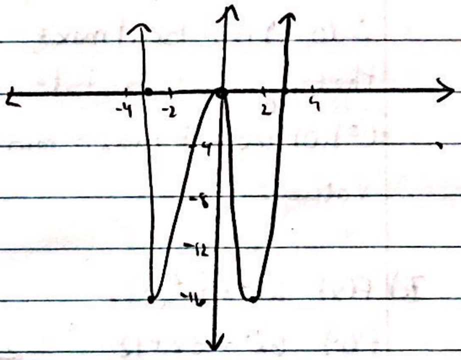
$(0,3)$ is a local min

$(2,27)$ is a local max

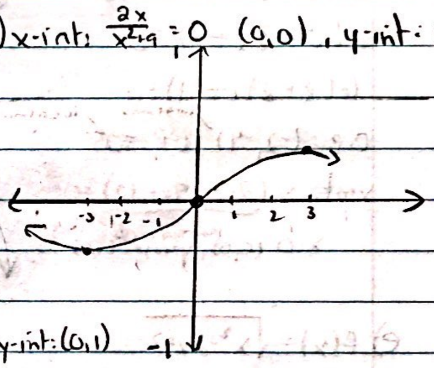
$h'(x)=0$ for both.

4a) x-int: $x^2(x^2-8)=0$ y-int: $(0,0)$

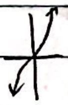
$(0,0),\ (\pm 2\sqrt{2},0) \rightarrow$ order 2



b) x-int: $\frac{2x}{x^2+9}=0$ $(0,0)$, y-int: $(0,0)$



5b) $g(t) = t^5+t^3$



$g'(t) = 5t^4+3t^2$

$t^2(5t^2+3)=0$

$t=0$

$\leftarrow t=\sqrt{-\frac{3}{5}}$ undefined!

↳ neither a max nor a min (no change wrt to ↑ and ↓)

c) $y=(x-5)^{\frac{1}{3}}$

$y' = \frac{1}{3(x-5)^{\frac{2}{3}}}$, $x \neq 5$ (for y')

- v. tangent at $x=5$

no max/min values on y.

- $(5,0)$ is a critical point on y, but is not a max/min.

d) $f(x) = (x^2-1)^{\frac{1}{3}}$

$f'(x) = \dfrac{2x}{3(x^2-1)^{\frac{1}{3}}}$ $x \neq \pm 1$ (v. tan, f.v undefined)

C.P. : $(0,-1)$, $(1,0)$, $(-1,0)$

Check $f'(-\frac{1}{2})$, $f'(\frac{1}{2})$

$f'(-\frac{1}{2}) = ⊕$ $f'(\frac{1}{2}) = ⊖$

∴ $(0,-1)$ is a local max + the tangent is horizontal

- $(\pm 1, 0)$ are not max or min values.

7a) $f(x) = -2x^2+8x+13$

$f'(x) = -4x+8$

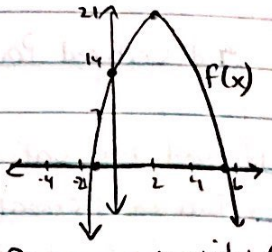
$-4(x-2) = 0$

C.P. $(2, 21)$ (max)

x-int: $-(2x^2-8x+13) = 0$

$x = \frac{8 \pm \sqrt{168}}{4}$ $(5.24, 0)$, $(-1.24, 0)$

y-int: $(0, 13)$



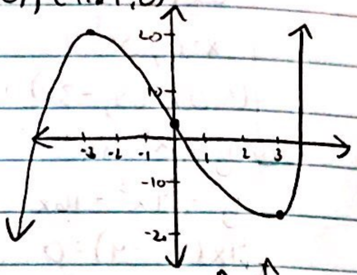
b) $f(x) = \frac{1}{3}x^3 - 9x + 2$

$f'(x) = x^2 - 9$

$(x+3)(x-3) = 0$ local max

C.P. : $(\overset{\text{local min}}{3}, -16)$, $(-3, 20)$

y-int: $(0, 2)$



7c) $f(x) = 2x^3+9x^2+12x$

$f'(x) = 6x^2+18x+12$

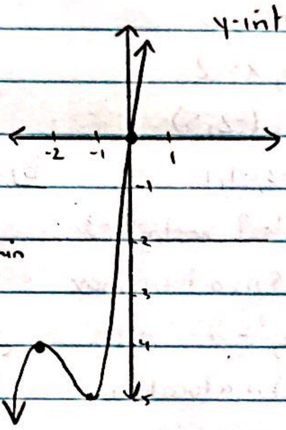
$6(x^2+3x+2) = 0$

$6(x+2)(x+1) = 0$

C.P. : $(-2, -4)$ local max, $(-1, -5)$ local min

x int: $x\underbrace{(2x^2+9x+12)}_{\text{no real roots}} = 0$

$x = 0$ $(0,0)$, y-int: $(0,0)$



d) $f(x) = -3x^3 - 5x$

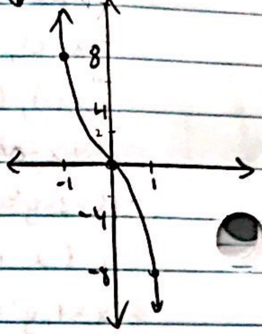
$f'(x) = -9x^2 - 5$

$9x^2 + 5 = 0$ no CPs!

$x = \pm\sqrt{-\frac{5}{9}}$

intercept: $(0,0)$

$f(-1) = 8$, $f(1) = -8$



e) $f(x) = \sqrt{x^2-2x+2}$

$f'(x) = \dfrac{2(x-1)}{2(x^2-2x+2)^{\frac{1}{2}}}$

$= \dfrac{x-1}{(x^2-2x+2)^{\frac{1}{2}}}$

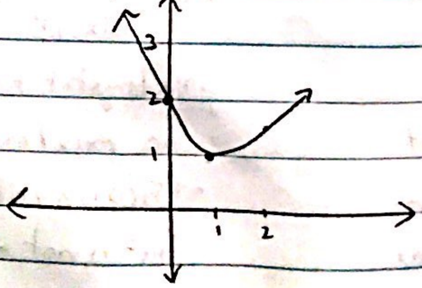
← no real roots, so no cusp

C.P: $(1, 1)$

$f'(0) = -\frac{1}{\sqrt{2}}$, $f'(2) = \frac{1}{\sqrt{2}}$

∴ Local min.

y-int: $(0, 2)$, x-int: none.



f) $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

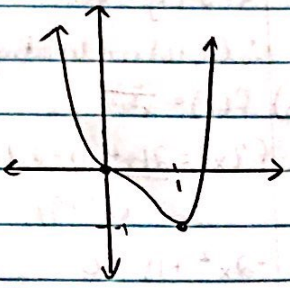
$12x^2(x-1) = 0$

CP: $(0,0)$, $(1,-1)$ ← local min

order 3, so not max or min

x-int: $x^3(3x-4) = 0$

$(\frac{4}{3}, 0)$



8. $f'(x) = (x+1)(x-2)(x+6)$

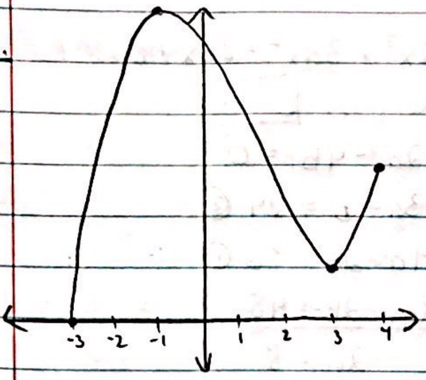
CN: $x = -1, 2, -6.$ (-1 local max, 2 local min, -6 local min)

$f'(-7) = ⊖$ $f'(-5) = ⊕$

$f'(-2) = ⊕$ $f'(0) = ⊖$

$f'(3) = ⊕$

9.



10. $y = ax^2 + bx + c$

y-int: $(0,1)$, so $\boxed{c=1}$

$y' = 2ax + b$

$6a + b = 0 \rightarrow b = -6a$

$12 = 9a + 3b + 1$

$9a + 3b = 11$

$9a - 18a = 11$

$\boxed{a = -\frac{11}{9}}$

$\boxed{b = \frac{22}{3}}$

11. $f(x) = x^2 + px + q$

$p + q = 4$

$f'(x) = 2x + p$

$f'(1) = 0$

$\therefore p = -2$

$q = 6$

$f(x) = x^2 - 2x + 6$

$f(0) = 6$

$f(2) = 6$ min because

$f(1) = 5$ ✓ a is ⊕.

12. $f(x) = x^3 - kx$

$f'(x) = 3x^2 - k \rightarrow x = \pm\sqrt{\frac{k}{3}}$

a) $k < 0$, no CNs

b) $k = 0$, one CN

c) $k > 0$, two CNs.

13. $g(x) = ax^3 + bx^2 + cx + d$ $g(0) = 0$, so $\boxed{d=0}$

① $8a + 4b + 2c = 4 \rightarrow 4a + 2b + c = 2$

$g'(x) = 3ax^2 + 2bx + c$ $g'(2) = 0$

② $12a + 4b + c = 0$ $g'(0) = 0$, so $\boxed{c=0.}$

$\therefore 4b = -12a$

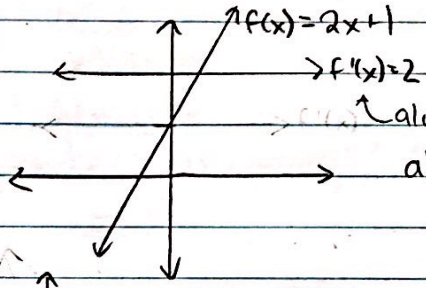
$b = -3a$ $4a + 2(-3a) + 0 = 2$

$\boxed{b = 3}$ $4a - 6a = 2$

$-2a = 2$

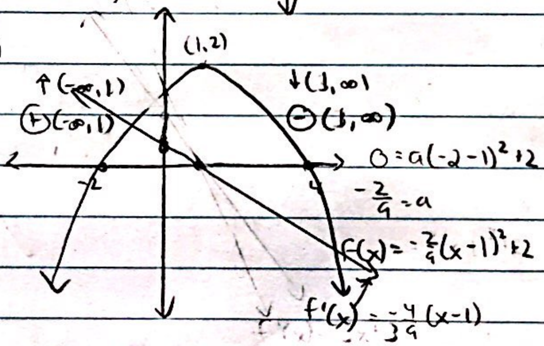
$\boxed{a = -1}$

14a)



↖always ⊕, and f(x) is always ↑.

14b)



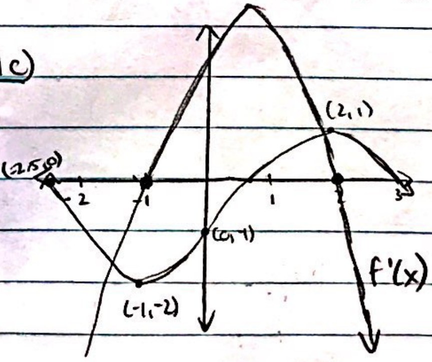
$0 = a(-2-1)^2 + 2$

$-\frac{2}{9} = a$

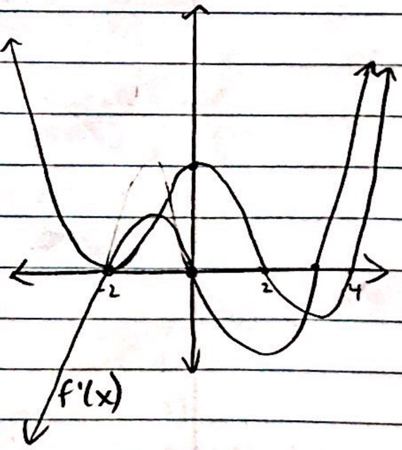
$f(x) = -\frac{2}{9}(x-1)^2 + 2$

$f'(x) = -\frac{4}{3}\frac{4}{9}(x-1)$

14c)



14d)



15. $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$

$f(0) = -9$, ∴ so $\boxed{d = -9}$

$-73 = 48 - 8a + 4b - 2c - 9$

$-112 = -8a + 4b - 2c$

$4a - 2b + \cancel{c} = 56$ ①

$f'(x) = 12x^3 + 3ax^2 + 2bx + c$

$f'(0) = 0$, so $\boxed{c = 0}$

$-96 + 12a - 4b = 0$

$3a - b = 24$ ②

$4a - 2b = 56$ ①

② × 2 $\quad 6a - 2b = 48$

⊖ $\quad -2a = 8$

$$\boxed{\begin{array}{l} a = -4 \\ b = -36 \end{array}}$$

4.3 Vertical and Horizontal Asymptotes

1a) $x = \pm 2,\ y = 1$
b) $x = 0,\ y = 0$

2. degree of top exactly one more than that of bottom → oblique
degree " " ≤ the degree of bottom → horizontal
bottom = 0 → vertical

3a) $f(x) = \frac{2x+3}{x-1} = \frac{2\cancel{x}(1+\frac{3}{2x})}{\cancel{x}(1-\frac{1}{x})}$

$= \frac{2(1+\frac{3}{2x})}{1-\frac{1}{x}}$

$\lim_{x\to\infty} f(x) = \frac{2[\lim_{x\to\infty}(1+\frac{3}{2x})]}{\lim_{x\to\infty}(1-\frac{1}{x})}$

$= \frac{2(1+0)}{1-0}$

$= 2$

$\lim_{x\to\infty} f(x) = \frac{2[\lim_{x\to-\infty}(1+\frac{3}{2x})]}{\lim_{x\to-\infty}(1-\frac{1}{x})}$

$= \frac{2(1+0)}{1-0}$

$= 2$

b) $f(x) = \frac{5x^2-3}{x^2+2}$

$= \frac{5\cancel{x^2}(1-\frac{3}{5x^2})}{\cancel{x^2}(1+\frac{2}{x^2})}$

$\lim_{x\to\infty} f(x) = \frac{5[\lim_{x\to\infty}(1-\frac{3}{5x^2})]}{\lim_{x\to\infty}(1-\frac{3}{5x^2})}$

$= \frac{5(1-0)}{1-0}$

$= 5$

$\lim_{x\to-\infty} f(x) = \frac{5[\lim_{x\to-\infty}(1-\frac{3}{5x^2})]}{\lim_{x\to-\infty}(1-\frac{3}{5x^2}}$

$= \frac{5(1+0)}{1-0}$

$= 5$

c) $f(x) = \frac{-5x^2+3x}{2x^2-5}$

$= \frac{-5\cancel{x^2}(1-\frac{3}{5x})}{2\cancel{x^2}(1-\frac{5}{2x^2})}$

$\lim_{x\to\infty} f(x) = \frac{-5[\lim_{x\to\infty}(1-\frac{3}{5x})]}{2\lim_{x\to\infty}(1-5/2x^2)}$

$= \frac{-5(1-0)}{2(1-0)}$

$= -5/2$

$\lim_{x\to-\infty} f(x) = \frac{-5[\lim_{x\to-\infty}(1-\frac{3}{5x})]}{2[\lim_{x\to-\infty}(1-\frac{5}{2x^2})]}$

$= \frac{-5(1-0)}{2(1-0)}$

$= -5/2$

d) $f(x) = \frac{2x^5-3x^2+5}{3x^4+5x-4}$

$= \frac{2x^{\cancel{5}}(1-\frac{3}{2x^3}+\frac{5}{2x^5})}{3\cancel{x^4}(1+\frac{5}{x^3}+\frac{4}{3x^4})}$

$\lim_{x\to\infty} f(x) = \frac{2[\lim_{x\to\infty} x \cdot \lim_{x\to\infty}(1-\frac{3}{2x^3}+\frac{5}{2x^5})]}{3[\lim_{x\to\infty}(1+\frac{5}{x^3}-\frac{4}{3x^4})]}$

$= \frac{2(\infty)(1-0+0)}{3(1+0-0)}$

$= \infty$

$\lim_{x\to-\infty} f(x) = \frac{2(-\infty)(1-0+0)}{3(1+0-0)}$

$= -\infty$

4a) $y = \frac{x}{x+5}$

- V.A @ $x = -5$

$\lim_{x\to-5^+} y = -\infty$ $(x = -4.99)$

$\lim_{x\to-5^-} y = \infty$ $(x = -5.01)$

b) $f(x) = \frac{x+2}{x-2}$

- V.A. @ $x = 2$

$\lim_{x\to2^+} f(x) = \infty$

$\lim_{x\to2^-} f(x) = -\infty$

c) $s = \frac{1}{(t-3)^2}$

- V.A @ $t = 3$
- large ⊕ on both sides.

d) $y = \frac{(x-3)(x+2)}{x-3}$

- hole @ $x = 3$

e) $f(x) = \frac{6}{(x+3)(x-1)}$

- V.A. @ $x = 1, x = -3$
- large ⊖ to the left of 1, large ⊕ to the right of 1.
- large ⊕ to the left of -3, large ⊖ to the right of -3.

f) $y = \frac{x^2}{(x+1)(x-1)}$

- V.A. @ $x = \pm 1$
- large ⊖ to the left of 1, large ⊕ to the right of 1
- large ⊕ to the left of -1, large ⊖ to the right of -1.

5a) $y = \frac{x}{x+4} = \frac{\cancel{x}}{x(1+\frac{4}{x})}$

$\lim_{x\to\infty} y = \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty}(1+\frac{4}{x})}$

$= \frac{1}{1+0}$

$\boxed{y=1}$

$\lim_{x\to-\infty} y = \frac{1}{1+0}$

$\boxed{y=-1}$

- When x is large ⊕, $f(x) < 1$, so below.
- When x is large ⊖, $f(x) > 1$, so above.

b) $f(x) = \frac{2x}{x^{\cancel{2}}(1-\frac{1}{x^2})}$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{2}{x(1-\frac{1}{x^2})}\right)$

$= \frac{\lim_{x\to\infty} 2}{(\lim_{x\to\infty} x)(\lim_{x\to\infty}(1-\frac{1}{x^2}))}$

$= \frac{2}{\infty(1-0)}$

$= 0$

$\lim_{x\to-\infty} f(x) = \frac{\lim_{x\to-\infty} 2}{(\lim_{x\to-\infty} x)(\lim_{x\to\infty}(1-\frac{1}{x^2}))}$

$= \frac{2}{-\infty}$

$= 0$

- When x is large ⊕, $f(x) > 0$, so above

When x is large ⊖, $f(x) < 0$, so below.

c) $g(t) = \frac{3\cancel{t^2}(1+\frac{4}{3t^2})}{\cancel{t^2}(1-\frac{1}{t^2})}$

$\lim_{t\to\infty} g(t) = \frac{3\lim_{x\to\infty}(1+\frac{4}{3t^2})}{\lim_{x\to\infty}(1-\frac{1}{t^2})}$

$= \frac{3(1+0)}{1-0}$

$= 3$

$\lim_{t\to-\infty} g(t) = \frac{3(1-0)}{1-0}$

$= 3$

- when t is large ⊕, $g(t) > 3$, so above
- when t is large ⊖, $g(t) > 3$, so above.

d) $y = \frac{3x^{\cancel{2}}(1-\frac{8}{3x}-\frac{7}{3x^2})}{\cancel{x}(1-4/x)}$

$\lim_{x\to\infty} y = \frac{(\lim_{x\to\infty} 3x)(\lim_{x\to\infty}(1-\frac{8}{3x}-\frac{7}{3x^2}))}{\lim_{x\to\infty}(1-4/x)}$

$= \frac{\infty(1+0)}{1+0}$

$= \infty$ } not equal, So no H.A.

$\lim_{x\to-\infty} = -\infty$ }

6a) $y = \frac{x-3}{x+5}$

$y' = \frac{(x+5)-(x-3)}{(x+5)^2}$

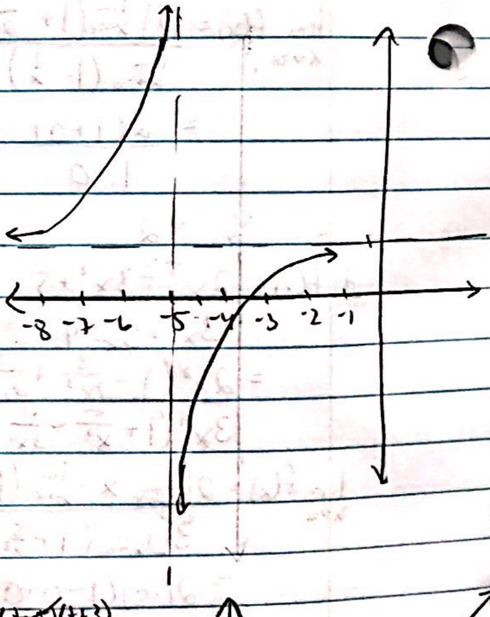
$= \frac{8}{}$ (no C.P's!)

always ⊕, so always ↑! → $(x+5)^2$

- V.A @ $x=-5$
- H.A @ $y=1$

When x is large ⊕, $y < 1$, so below.

When x is large ⊖, $y > 1$, so above.

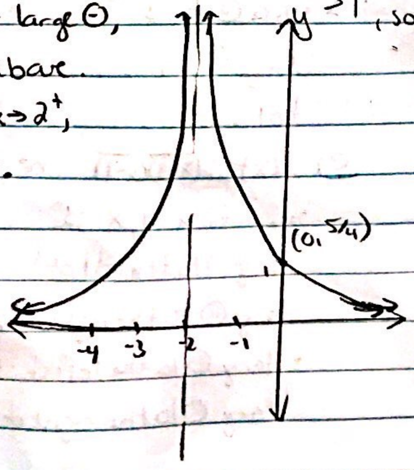


6b) $f(x) = \frac{5}{(x+2)^2}$

V.A @ $x=-2$, H.A. @ $y=0$

When x is large ⊕ or large ⊖, $f(x) > 0$, so above.

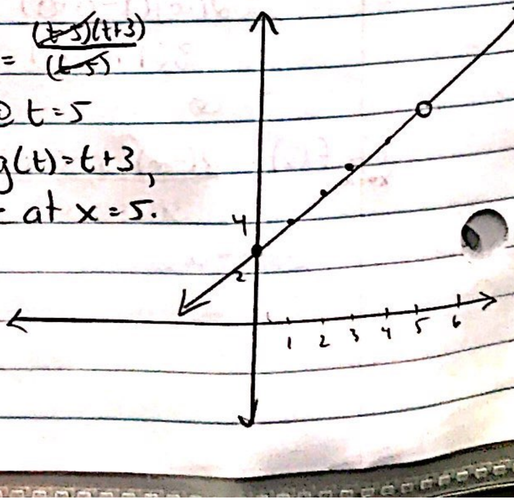
When $x \to 2^-$, or $x \to 2^+$, $f(x)$ is large ⊕.



6c) $g(t) = \frac{\cancel{(t-5)}(t+3)}{\cancel{(t-5)}}$

∴ Hole @ $t=5$

Line $g(t) = t+3$, hole at $x=5$.



★ I accidentally skipped part d → It's on the next page after #9!

7a) $x-3\overline{)3x^2-2x-17}$ quotient $3x+7$

$3x^2-9x$

$7x-17$

$7x-21$

4

$\therefore$ Oblique asymptote

$y=3x+7$

b) $2x+3\overline{)2x^2+9x+2}$ quotient $x+3$

$2x^2+3x$

$6x+2$

$6x+9$

-7

$\therefore$ Oblique asymptote

$y=x+3$

c) $x^2+2x\overline{)x^3+0x^2+0x-1}$ quotient $x-2$

x^3+2x^2

$-2x^2+0x$

$-2x^2-4x$

$\boxed{R|4x-1}$

$\therefore$ Oblique asymptote

$y=x-2$

d) $x^2-4x+3\overline{)x^3-x^2-9x+15}$ quotient $x+3$

x^3-4x^2+3x

$3x^2-12x+15$

$3x^2-12x+9$

6

$\therefore$ Oblique asymptote

$y=x+3$

8a) $f(x)=\dfrac{3x^2-2x-17}{x-3}$

307.04 307

As x gets very large ⊕,

$f(x)>3x+7$ but approaches it, so it come from above.

As x gets very large ⊖,

$f(x)<3x+7$, so below.

b) $f(x)=\dfrac{2x^2+9x+2}{2x+3}$ 10.2.96

As x gets large ⊕,

$f(x)<x+3$, so below.

As x gets large ⊖,

$f(x)>x+3$, so above.

9a) $f(x)=\dfrac{3x-1}{x+5}$

V.A. @ $x=-5$

H.A. @ $y=3$

As $x\to-5^-$, $f(x)\to\infty$

As $x\to-5^+$, $f(x)\to-\infty$

b) $g(x)=\dfrac{x^2+3x-2}{(x-1)^2}$

V.A. @ $x=1$

$$\lim_{x\to\infty}g(x)=\lim_{x\to\infty}\left[\frac{\cancel{x^2}\left(1+\frac{3}{x}-\frac{2}{x^2}\right)}{\cancel{x^2}\left(1-\frac{2}{x}+\frac{1}{x^2}\right)}\right]$$

$$=1$$

H.A @ $y=1$

As $x\to1^-$, $g(x)\to\infty$

As $x\to1^+$, $g(x)\to\infty$

c) $h(x)=\dfrac{(x+3)(x-2)}{(x+2)(x-2)}$

Hole @ $x=2$

V.A. @ $x=-2$

H.A. @ $y=1$

As $x\to-2^-$, $h(x)\to-\infty$

As $x\to-2^+$, $h(x)\to\infty$

d) $m(x)=\dfrac{(5x+2)(x-1)}{x-2}$

VA @ $x=2$

As $x\to2^-$, $m(x)\to-\infty$

As $x\to2^+$, $m(x)\to\infty$

$\lim_{x\to\infty}m(x)\neq\lim_{x\to-\infty}m(x)$,

so no H.A.

$\boxed{6d)}$ $y=\dfrac{(x+2)(-2x+3)}{x(x-3)}$ V.A. @ $x=0$, $x=3$

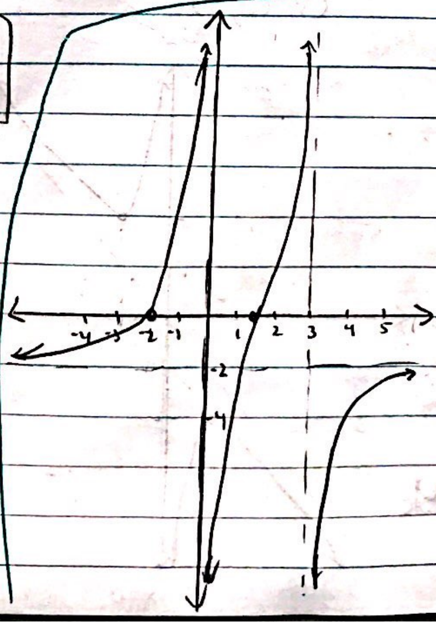
$$y=\frac{-2x^2-x+6}{x^2-3x}=\frac{-2\cancel{x^2}\left(1+\frac{1}{2x}-\frac{6}{2x^2}\right)}{\cancel{x^2}\left(1-\frac{3}{x}\right)}$$

$$\lim_{x\to\infty}y=\frac{-2\left[\lim_{x\to\infty}\left(1+\frac{1}{2x}-\frac{3}{x^2}\right)\right]}{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)}$$

$$=\frac{-2(1+0-0)}{1-0}$$

$$=-2$$ H.A @ $y=-2$

$\lim_{x\to-\infty}y=-2.$



x-int: $(-2,0)$, $(\frac{3}{2},0)$

As $x\to0^-$, $y\to\infty$

As $x\to0^+$, $y\to-\infty$

As $x\to3^-$, $y\to\infty$

As $x\to3^+$, $y\to-\infty$

When x is large ⊕, $y<-2$, so below

When x is large ⊖, $y>-2$, so above

10a) $f(x) = \frac{3-x}{2x+5}$

① VA @ $x = -5/2$

As $x \to -\frac{5}{2}^-$, $f(x) \to -\infty$

As $x \to -\frac{5}{2}^+$, $f(x) \to \infty$

② x-int: $(3, 0)$

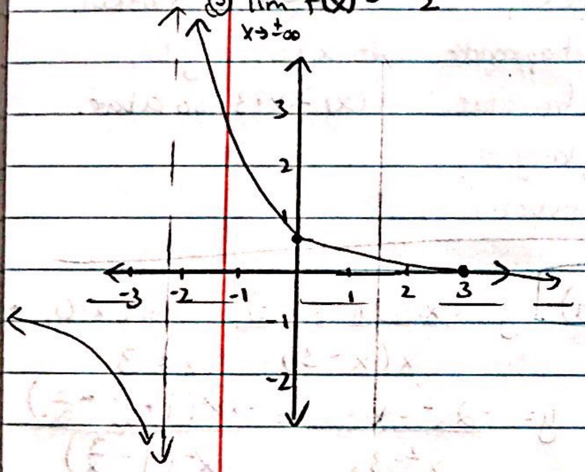
y-int: $(0, 3/5)$

③ $f'(x) = \frac{-(2x+5) - 2(3-x)}{(2x+5)^2}$

Can't equal zero, $\rightarrow = \frac{-11}{(2x+5)^2}$

so no C.P.'s ⑥ Always ↓!

⑤ $\lim_{x \to \pm\infty} f(x) = -\frac{1}{2}$



b) $h(t) = 2t^3 - 15t^2 + 36t - 10$

① $D = \{x \in \mathbb{R}\}$, no asymptotes.

② y-int: $(0, -10)$

③ $h'(t) = 6t^2 - 30t + 36$

$t^2 - 5t + 6 = 0$

$(t-3)(t-2) = 0$

C.P.'s at $(3, 17), (2, 18)$

④ $h'(2.99) = ⊖$, $h'(3.01) = ⊕$, so min.

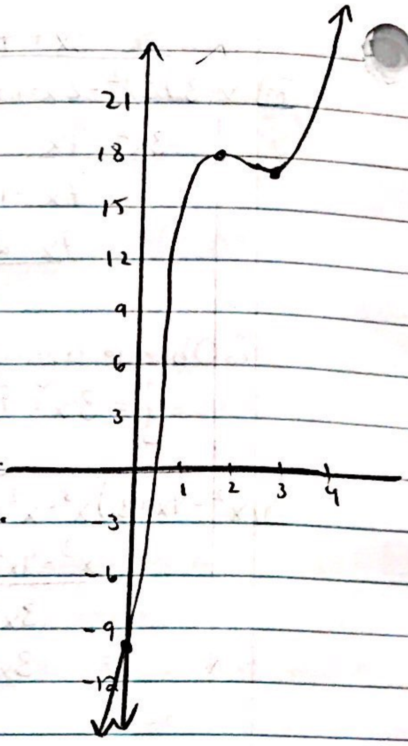
$h'(2.01) = ⊖$ $h'(1.99) = ⊕$, so max.

⑤ As ~~t~~ $t \to \infty$, $h(t) \to \infty$

As $t \to -\infty$, $h(t) \to -\infty$

⑥ ↑ $(-\infty, 2), (3, \infty)$

↓ $(2, 3)$



c) $y = \frac{20}{x^2+4}$

① no V.A. always ⊕.

② y-int: $(0, 5)$

No x-int.

③ $y' = \frac{-20(2x)}{(x^2+4)^2}$

$-40x = 0$

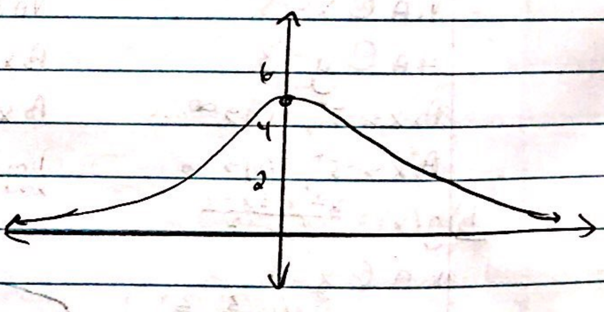
$x = 0$

CP: $(0, 5)$

④ y' when $x<0$ is ⊕, y' when $x>0$ is ⊖, so max.

⑤ H.A. @ $y = 0$, Always above.

⑥ ↑ $(-\infty, 0)$, ↓ $(0, \infty)$



d) $s(t) = t + \frac{1}{t}$

① $t \neq 0$, V.A @ $t = 0$

As $t \to 0^-$, $s(t) \to -\infty$.

As $t \to 0^+$, $s(t) \to \infty$

② no intercepts

③ $s'(t) = 1 - \frac{1}{t^2}$

$t^2 - 1 = 0$

$t = \pm 1$ min max

CPs @ $(1, 2), (-1, -2)$

④ ~~t~~ <1, $s(t) \to ⊖$; ~~t~~ >1, $s(t) \to ⊕$

$t < -1$, $s(t) \to +$; $t > -1$, $s(t) \to ⊖$

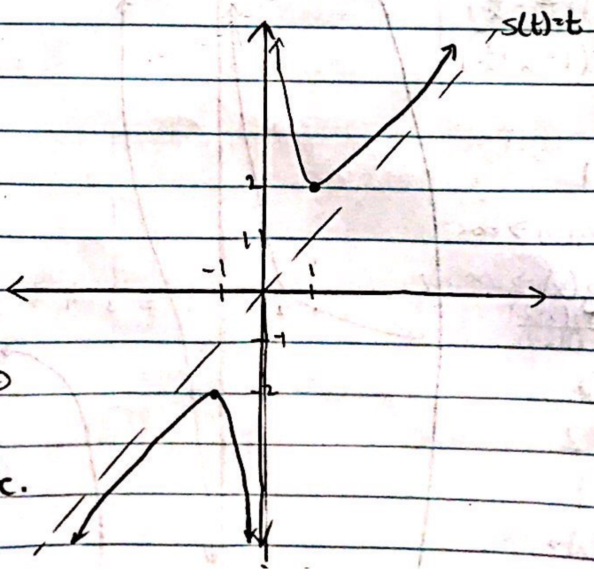
$$t\overline{)t^2+1}\quad \text{quotient } t,\ \ t^2,\ \ 0+1$$

⑤ $s(t) = \frac{t^2+1}{t}$

$s(t) = t$ ⇐ oblique asymptote.

⑥ ↑ $(-\infty, -1), (1, \infty)$

↓ $(-1, 0)$



e) $g(x) = \frac{2x^2+5x+2}{x+3} = \frac{(2x+1)(x+2)}{x+3}$

① V.A. @ $x = -3$

As $x \to -3^-$, $g(x) \to -\infty$

As $x \to -3^+$, $g(x) \to \infty$

② x-int: $(-\frac{1}{2}, 0)$, $(-2, 0)$

y-int: $(0, \frac{2}{3})$

③ $g'(x) = \frac{(4x+5)(x+3) - 2x^2-5x-2}{(x+3)^2}$

$= \frac{2x^2+12x+13}{(x+3)^2}$

$2x^2+12x+13 = 0$

$x = \frac{-12 \pm \sqrt{40}}{4}$

$= \frac{-6 \pm \sqrt{10}}{2}$ (-1.4)

④ CPs @ $(\frac{-6+\sqrt{10}}{2}, -5.6)^{min}$

and $(\frac{-6-\sqrt{10}}{2}, -13.3)^{max}$
-(4.6)

⑤
$$\begin{array}{r} 2x-1 \\ x+3 \overline{)2x^2+5x+2} \\ \underline{2x^2+6x} \\ -x+2 \\ \underline{-x-3} \\ 5 \end{array}$$

Oblique asymptote

$y = 2x-1$

⑥ ↑ $(-\infty, \frac{-6-\sqrt{10}}{2})$, $(\frac{-6+\sqrt{10}}{2}, \infty)$

↓ $(\frac{-6-\sqrt{10}}{2}, 0)$, $(0, \frac{-6+\sqrt{10}}{2})$

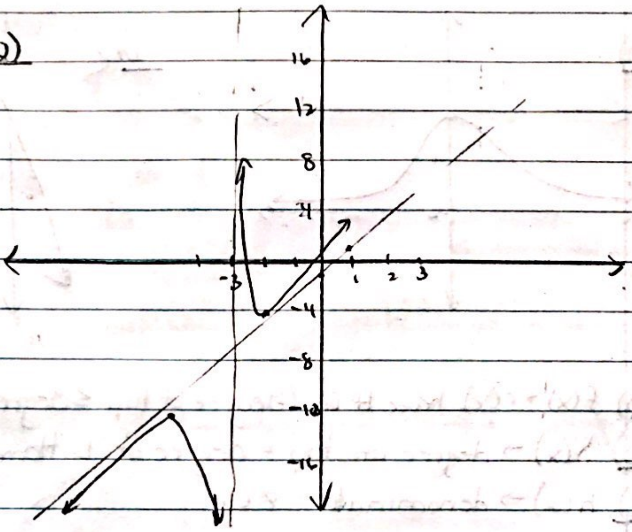
11a) $y = \frac{a\cancel{x}(1+\frac{b}{ax})}{c\cancel{x}(1+\frac{d}{cx})}$

b) $x = -\frac{d}{c}$

$\lim_{x\to\infty} y = \frac{a\left[\lim_{x\to\infty}(1+\frac{b}{ax})\right]}{c\left[\lim_{x\to\infty}(1+\frac{c}{dx})\right]}$

$= \frac{a(1+0)}{c(1+0)}$

$\boxed{y = a/c}$

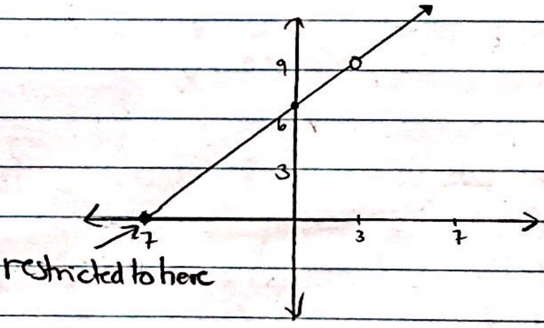


f) $s(t) = \frac{t^2+4t-21}{(t-3)}$, $t \geq -7$ $s(t) = \frac{(t+7)\cancel{(t-3)}}{\cancel{t-3}}$

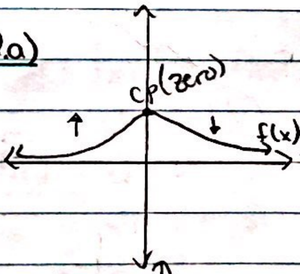
① Hole @ ~~x~~t = 3, no V.A.

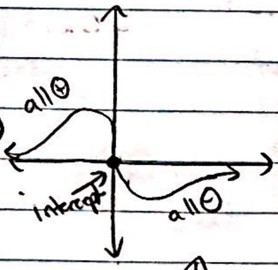
② ~~x~~t-int: (-7, 0), y-int: (0, 7)

Line w/ hole @ ~~x~~t = 7, slope of 1.

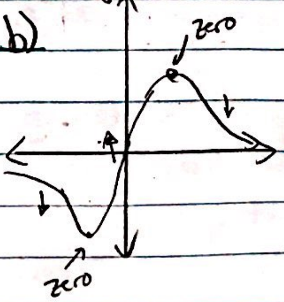


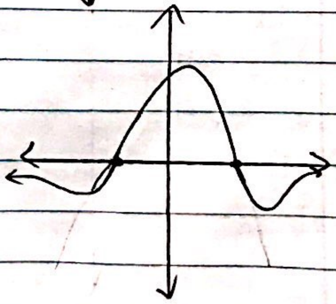
12a)



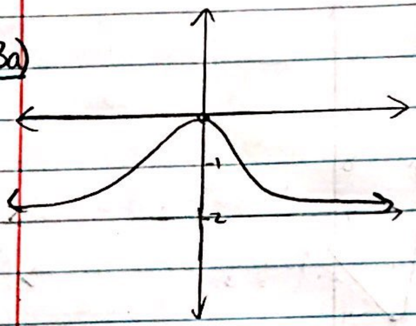


b)

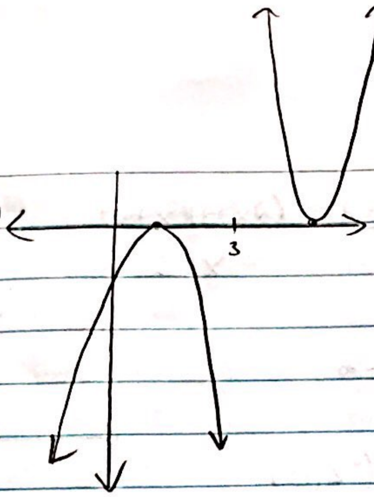




13a)



b)



14a) f(x) + r(x) have H. A. (degree of top ≤ degree of bottom)

b) h(x) → degree of top = degree of bottom + 1.

c) h(x) → denominater ≠ 0.

4.4 Concavity + Points of Inflection

1a) ⊖, ⊖, ⊕, ⊕

b) ⊖, ⊖, ⊕, ⊖

2a) $y = x^3 - 6x^2 - 15x + 10$

$y' = 3x^2 - 12x - 15$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5, x = -1$

$\text{CP's}: \underset{\text{min}}{(5, -90)}, \underset{\text{max}}{(-1, 18)}$

$y'' = 6x - 12$

$6(x-2) = 0$

*Concavity changes at $x = 2$.

$y'' =$ ⊖ when $x = -1$ $\cap$

$y'' =$ ⊕ when $x = 5$ $\cup$

3a) P.O.I. at $(2, -36)$

2b) $y = \frac{25}{x^2+48}$ · H.A. @ $y = 0$, all y-values will be ⊕.

$y' = \frac{-25(2x)}{(x^2+48)^2} = \frac{-50x}{(x^2+48)^2}$

CP: $(0, 25/48)$ ← max.

$y'' = \frac{-50(x^2+48)^2 + 50x[2(x^2+48)(2x)]}{(x^2+48)^4}$

$= \frac{-50(x^2+48)[x^2+48-4x^2]}{(x^2+48)^4}$

$= \frac{-50(x^2+48)(-3x^2+48)}{(x^2+48)^4}$

$= \frac{-50(-3x^2+48)}{(x^2+48)^3}$

$y'' =$ ⊖ when $x = 0$, so $\cap$

3b) $-3x^2 + 48 = 0$ P.O.I.'s @ $(-4, \frac{25}{64})$ and $(4, \frac{25}{64})$

$x^2 - 16 = 0$

$x = \pm 4$

2c) $s = t + t^{-1}$

$s'(t) = 1 - t^{-2}$

$= \frac{t^2-1}{t^2}$

$(t+1)(t-1) = 0$

$\text{C.Ps} @ \underset{\text{min}}{(1, 2)} + \underset{\text{max}}{(-1, -2)}$

$s''(t) = -2t^{-3}$

$= \frac{2}{t^3}$

$s''(1) =$ ⊕ $\cup$ $s''(-1) =$ ⊖ $\cap$

3c) $\frac{2}{t^3} = 0$ ← not possible

∴ no points of inflection.

2d) $y = (x-3)^3 + 8$

$y' = 3(x-3)^2$

CP @ $(3, 8)$ ← neither

$y'' = 6(x-3)$

$y'' = 0$ when $x = 3$

3d) Point of inflection at $(3, 8)$.

4b) $g(x) = x - \frac{1}{x}$

$g'(x) = 1 + \frac{1}{x^2}$

$g''(x) = -\frac{2}{x^3}$

$g''(-1) = 2$

∴ The graph is above the tangent.

4a) $f(x) = 2x^3 - 10x + 3$

$f'(x) = 6x^2 - 10$

$f''(x) = 12x$

$f''(2) = 24$ (⊕)

∴ The graph is above the tangent (c.u.) at this point.

4d) $s(t) = \frac{2t}{t-4}$

$s'(t) = \frac{2(t-4) - 2t(1)}{(t-4)^2}$

$= \frac{-8}{(t-4)^2}$

$s''(t) = \frac{8(2(t-4))}{(t-4)^4}$

$= \frac{16}{(t-4)^3}$

$s''(-2) = -\frac{2}{27}$

∴ The graph is below the tangent.

4c) $p(\omega) = \frac{\omega}{\sqrt{\omega^2+1}}$

$p'(\omega) = \frac{\sqrt{\omega^2+1} - \omega\left(\frac{1}{2}(\omega^2+1)^{-1/2}(2\omega)\right)}{\omega^2+1}$

$= \frac{\omega^2+1-\omega^2}{(\omega^2+1)^{3/2}}$

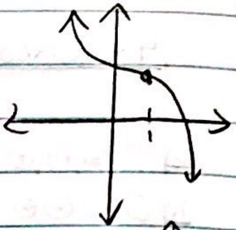
$= \frac{1}{2(\omega^2+1)^{3/2}}$

$p''(\omega) = \frac{-\left[3(\omega^2+1)^{1/2}(2\omega)\right]}{4(\omega^2+1)^3}$

$= \frac{-3\omega\sqrt{\omega^2+1}}{(\omega^2+1)^3}$

$p''(3) = -\frac{9\sqrt{10}}{1000}$

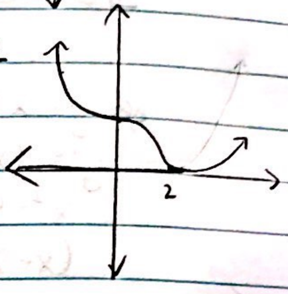
f(x) is cubic



5a) C.U. when $x<1$

C.D. when $x>1$

P.O.I. @ $x=1$



b) C.U. when $x<0$, $x>2$

C.D. when $0<x<2$

P.O.I.'s @ $x=0$, $x=2$

f(x) is quartic.

6. ⊕ → C.U, so min.

⊖ → C.D, so max.

7. Use the second derivative test to classify critical points.

8a) $f(x) = x^4+4x^3$ ← no asymptotes

$f'(x) = 4x^3+12x^2$

$4x^2(x+3)=0$

$\boxed{\text{C.P.s } (\overset{\text{POI}}{0,0}) \text{ and } (\overset{\text{max}}{-3},-27)}$

$f''(x) = 12x^2+24x$

$12x(x+2)=0$

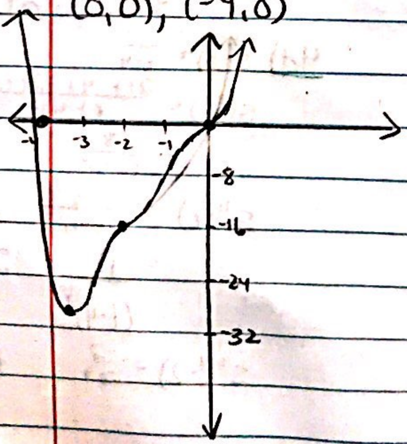
$\boxed{\text{Point of inflection at } (0,0), (-2,-16)}$

$f''(-3) = ⊖$, so ∩

Intercepts:

$x^3(x+4)=0$

$(0,0), (-4,0)$



$\frac{6}{-\frac{27}{16}}$

b) $g(\omega) = \frac{4\omega^2-3}{\omega^3}$ → $\boxed{\text{V.A. @ } \omega=0}$

$\boxed{\text{Intercepts: } (\frac{\sqrt{3}}{2},0), (-\frac{\sqrt{3}}{2},0)}$

$g'(\omega) = \frac{8\omega(\omega^3)-3\omega^2(4\omega^2-3)}{\omega^6}$

$= \frac{8\omega^4-12\omega^4+9\omega^2}{\omega^6}$

$= \frac{-4\omega^2+9}{\omega^4}$

$-(2\omega+3)(2\omega-3)=0$

$\boxed{\text{CPs @ } (-\frac{3}{2}, -\frac{16}{9}) \text{ and } (\frac{3}{2}, \frac{16}{9})}$ min max

$g''(\omega) = \frac{-8\omega(\omega^4)-4\omega^3(-4\omega^2+9)}{\omega^8}$

$= \frac{-8\omega^5+16\omega^5-36\omega^3}{\omega^8}$

$= \frac{8\omega^2-36}{\omega^5}$

$0 = \omega^2 = \frac{36}{8}$

$\omega = \pm\frac{6}{2\sqrt{2}}$

$= \pm\frac{3}{\sqrt{2}}$

$\boxed{\text{POIs @ } (\frac{3}{\sqrt{2}}, \frac{-8\sqrt{2}}{9}) \text{ and } (-\frac{3}{\sqrt{2}}, \frac{8\sqrt{2}}{9})}$

$g''(\frac{3}{2}) = ⊖$ ∩ $g''(-\frac{3}{2}) = ⊕$ ∪

Horizontal asymptote:

$\lim_{x\to\infty} g(\omega) = \frac{\lim_{x\to\infty}\left[4\left(1-\frac{3}{4\omega^2}\right)\right]}{\lim_{x\to\infty}\omega}$

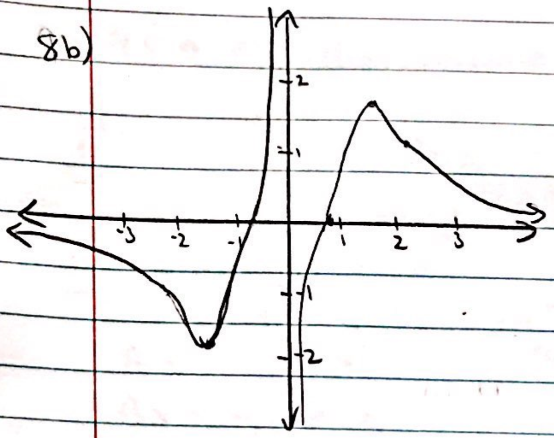
$= 0$

$\boxed{y=0}$

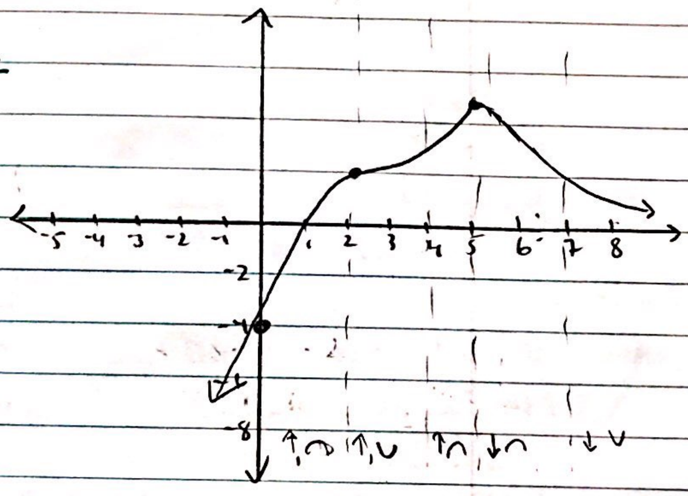
$\lim_{x\to 0^-} g(\omega) = \infty$

$\lim_{x\to 0^+} g(\omega) = -\infty$

8b)



9.



- $f(x)$ ↑ $(-\infty, 2), (2, 5)$
- $f(x)$ ↓ $(5, \infty)$
- C.Ps @ $x = 2$ ← inflection, $x = 5$ ← max on $f(x)$
- C.U $(2, 4), (7, \infty)$
- C.D. $(-\infty, 2), (4, 7)$

10. $f(x) = ax^3 + bx^2 + c$

$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

$f'(2) = 0$ $\qquad$ $f''(1) = 0$

$12a + 4b = 0$ $\qquad$ $6a + 2b = 0$

$\therefore b = -3a$ $\qquad$ $\therefore b = -3a.$

$f(1) = 5,$ $\qquad$ $f(2) = 11$

$a + b + c = 5$ $\qquad$ $8a + 4b + c = 11$

$a - 3a + c = 5$ $\qquad$ $8a - 12a + c = 11$

$-2a + c = 5$ $\qquad$ $-4a + c = 11$

$c = 5 + 2a$ $\qquad$ $-4a + 5 + 2a = 11$

$c = -1$ $\qquad$ $-2a = 6$

$f(x) = -3x^3 + 9x^2 - 1$ $\qquad$ $a = -3$

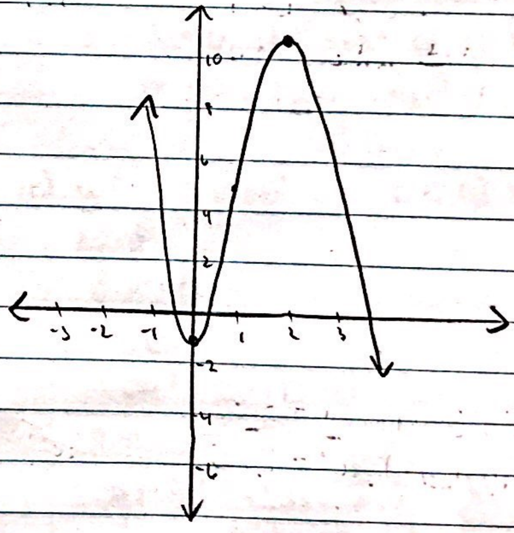
$b = 9$

$f'(x) = -9x^2 + 18x$

$-9x(x - 2) = 0$

$x = 0, x = 2$

CPs @ $\overset{\text{max}}{(2, 11)}, \overset{\text{min}}{(0, -1)}$

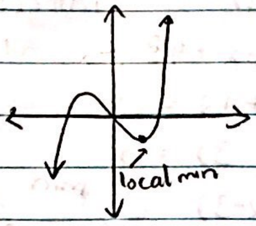


$f''(x)$ is a linear fcn,

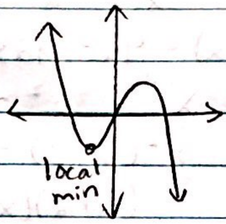
$\therefore$ so only one POI.

4.5 An Algorithm for Curve Sketching

1.



OR



As $x \to \infty, y \to \infty$
As $x \to -\infty, y \to -\infty$

As $x \to \infty, y \to -\infty$
As $x \to -\infty, y \to \infty$

2. Degree 3 → 1 local max, 1 local min
Degree 4 → 2 local min + 1 local max or 2 local max + 1 local min
Degree n → (n-1) extreme values.
BECAUSE $f(x) = x^{n} \to f'(x) = x^{(n-1)}$

3a) $y = \frac{x}{(x+3)(x+1)}$

$x = -3, x = -1$

b) $y = \frac{5x-4}{x^2-6x+12}$

$x = \frac{6 \pm \sqrt{36-48}}{2}$ ← doesn't exist

$\therefore$ no vertical asymptotes

c) $y = \frac{3x+2}{(x-3)^2}$

$x = 3$

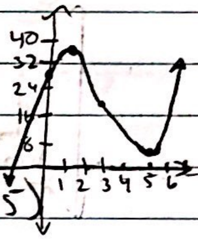
4a) $y = x^3 - 9x^2 + 15x + 30$
- no asymptotes
- As $x \to \infty, y \to \infty$, As $x \to -\infty, y \to -\infty$.
- y-intercept (0,30)

$y' = 3x^2 - 18x + 15$
$\therefore x^2 - 6x + 5 = 0$
$(x-5)(x-1) = 0$

min max

Critical Points: (5,5), (1,37)

↑ $x \in (-\infty, 5)$, $(1, \infty)$, ↓ $x \in (1, 5)$



$y'' = 6x - 18$
$\therefore x - 3 = 0$
$x = 3$

Point of inflection at (3,21)

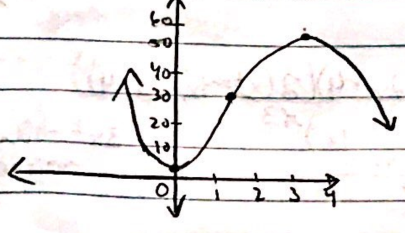
4b) $f(x) = -4x^3 + 18x^2 + 3$
- no asymptotes
- as $x \to \infty, y \to -\infty$
- as $x \to -\infty, y \to \infty$
- y-int: (0,3)

$f'(x) = -12x^2 + 36x$
$\therefore -12x(x-3) = 0$

min max

Critical points: (0,3), (3,57)
$f''(x) = -24x + 36$
$\therefore -12(2x-3) = 0$
$x = \frac{3}{2}$

Point of Inflection: $(\frac{3}{2}, 30)$



4c) $y = 3 + \frac{1}{(x+2)^2}$ V.A @ $x = -2$, y-int: $(0, 3\frac{1}{4})$

x-int

$\frac{3(x^2+4x+4)+1}{(x+2)^2} = 0$

$3x^2 + 12x + 13 = 0$ $x = \frac{144 \pm \sqrt{-12}}{6}$ no x-int!

H.A.

$y = \frac{3\cancel{x^2}(1 + \frac{4}{3x} + \frac{5}{3x^2})}{\cancel{x^2}(1 + \frac{4}{x} + \frac{4}{x^2})}$ $\lim_{x \to \pm\infty} y = \frac{3(1+0+0)}{(1+0+0)}$ $\boxed{y = 3}$

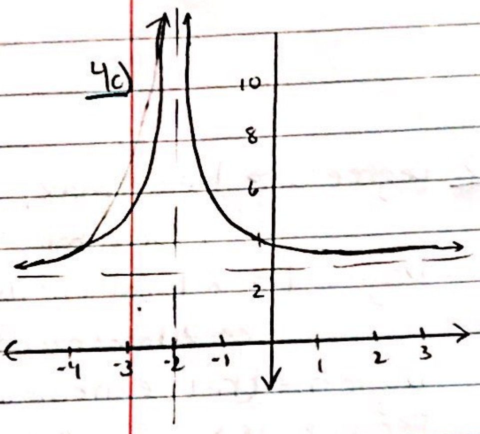
$y' = \frac{-2(x+2)}{(x+2)^4} = \frac{-2}{(x+2)^3}$

y' is ⊕ when $x < -2$ and ⊖ when $x > -2$
$\therefore$ ↑ $(-\infty, -2)$, ↓ $(-2, \infty)$

$y'' = \frac{2(3(x+2)^2)}{(x+2)^6} = \frac{6}{(x+2)^4}$ no points of inflection, always ⊕, so always C.U.

(Sketch on back!)

4c)



d) $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

$= x(x^3 - 4x^2 - 8x + 48)$

- no asymptotes
- As $x \to \pm\infty$, $f(x) \to \infty$.
- intercept (0,0)

$f'(x) = 4x^3 - 12x^2 - 16x + 48$

$\therefore x^3 - 3x^2 - 4x + 12 = 0$

$x^2(x-3) - 4(x-3) = 0$

$(x+2)(x-2)(x-3) = 0$

min max min

Critical points: $(-2, -80)$, $(2, 48)$, $(3, 45)$

$f''(x) = 12x^2 - 24x - 16$

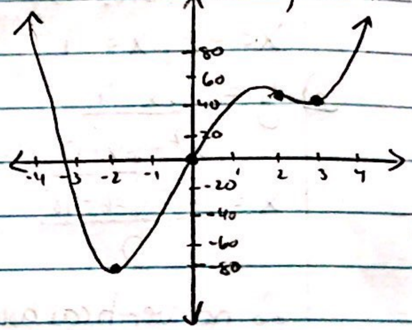
$\therefore 3x^2 - 6x - 4 = 0$

$x = \frac{6 \pm 2\sqrt{21}}{6} = \frac{3 \pm \sqrt{21}}{3}$

Points of inflection at:

$(\sim 2.5, 46.6)$

$(\sim 0.5, -25.4)$



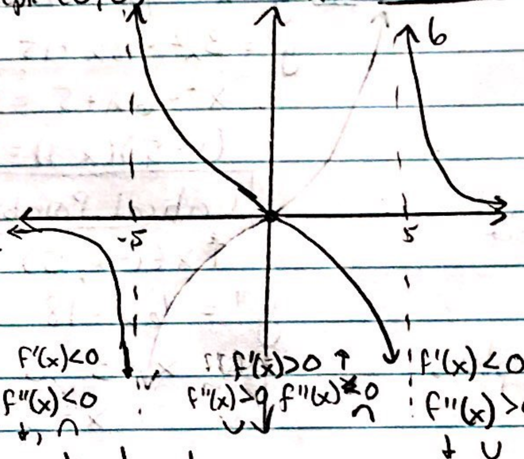
e) $y = \frac{2x}{x^2-25}$ V.A. $x = \pm 5$

$\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty}\left(\frac{2x}{x^2(1-\frac{25}{x^2})}\right)$ Intercept (0,0)

$= \lim\limits_{x\to\infty}\left(\frac{2}{x(1+\frac{25}{x^2})}\right)$

$\boxed{y = 0}$ ⟸ H.A.

$\lim\limits_{x\to-\infty} y = 0$



$y' = \frac{2(x^2-25) - 4x^2}{(x^2-25)^2}$

$= \frac{-2x^2 - 50}{(x^2-25)^2}$

$-2x^2 - 50 \neq 0$, so no critical points.

$y'' = \frac{-4x(x^2-25)^{\cancel{2}} - (-2x^2-50)(2x(2\cancel{(x^2-25)}))}{(x^2-25)^{\cancel{4}3}}$

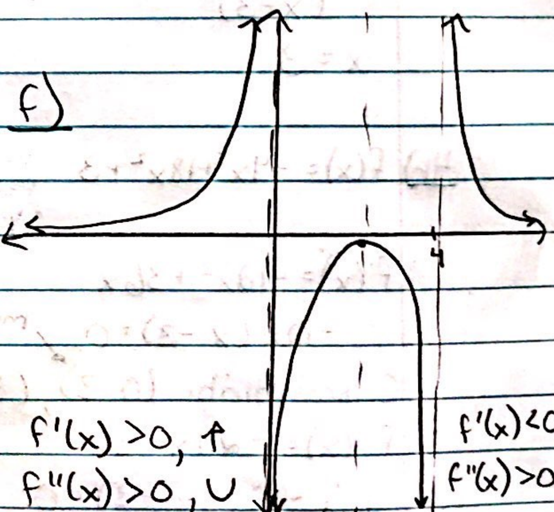
$= \frac{-4x^3 + 8x^3 + 200x}{(x^2-25)^3}$

$= \frac{4x^3 + 200x}{(x^2-25)^3}$

$\therefore 4x(x^2+50) = 0$

Point of inflection at (0,0)

f)



f) $f(x) = \frac{1}{x(x-4)}$ V.A @ $x=0$, $x=4$

$f'(x) = \frac{-(2x-4)}{(x^2-4x)^2}$ H.A @ $f(x) = 0$

y-int: $(0, -\frac{1}{4})$

$= \frac{-2x+4}{(x^2-4x)^2}$

max

C.P. at $(2, -\frac{1}{4})$

$f''(2) < 0$, so ∩

$f''(x) = \frac{-2(x^2-4x)^{\cancel{2}} - (-2x+4)(2\cancel{(x^2-4x)}(2x-4))}{(x^2-4x)^{\cancel{4}3}}$

$= \frac{-2x^2+8x - (-2x+4)(4x-8)}{(x^2-4x)^3}$

$= \frac{6x^2 - 24x + 32}{(x^2-4x)^3}$

$\therefore 3x^2 - 12x + 16 = 0$

no real roots, so no points of inflection

g) $y = \frac{6x^2-2}{x^3}$

$= \frac{6x^2(1-\frac{3}{x^2})}{x^3}$

$\lim_{x\to\pm\infty} y = \frac{6\lim_{x\to\pm\infty}(1+\frac{3}{x^2})}{\lim_{x\to\pm\infty} x}$

$= 0$

$\boxed{\therefore \text{H.A.} @ y=0}$

$\boxed{\text{V.A.} @ x=0}$ (no y-int.)

$6x^2 = 2$

$x^2 = \frac{1}{3}$

$\boxed{\therefore \text{x-int} @ (\pm\frac{1}{\sqrt{3}}, 0)}$

$y' = \frac{12x(x^3) - (6x^2-2)(3x^2)}{x^6}$

$= \frac{12x^{\cancel{4}2} - 18x^{\cancel{4}2} + 6\cancel{x}}{x^{\cancel{6}4}}$

$= \frac{-6x^2+6}{x^4}$

$-6x^2+6=0$

$x = \pm 1$

$\boxed{\text{CP's at } (1,4) \text{ and } (-1,-4)}$

$y'' = \frac{-12x(x^4) - (-6x^2+6)(4x^3)}{x^8}$

$= \frac{-12x^5 + 24x^5 - 24x^3}{x^8}$

$= \frac{12x^5 - 24x^3}{x^8}$

$= \frac{12x^2 - 24}{x^5}$

$\boxed{\text{Point of inflection at } (\sqrt{2}, \frac{5}{\sqrt{2}}) \text{ and } (-\sqrt{2}, \frac{-5}{\sqrt{2}})}$

As x→0⁻, y→∞

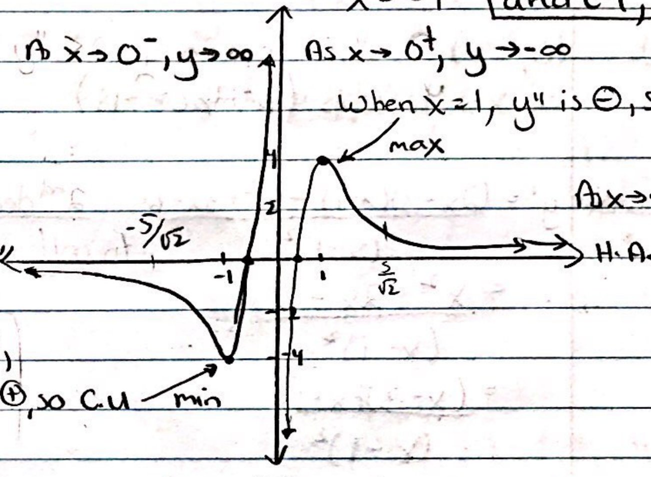
As x→0⁺, y→-∞

When x=1, y'' is ⊖, so CD

As x→-∞, y→0

As x→∞, y→0

When x=-1, y'' is ⊕, so C.U



h) $f(x) = \frac{x+3}{x^2-4} = \frac{x+3}{(x+2)(x-2)}$

Asymptotes: V.A. @ $x = \pm 2$

$f(x) = \frac{\cancel{x}(1+\frac{3}{x})}{x^{\cancel{2}}(1-\frac{4}{x^2})}$

$\lim_{x\to\pm\infty} f(x) = \frac{\lim_{x\to\pm\infty}(1+\frac{3}{x})}{\lim_{x\to\pm\infty} x\left[\lim_{x\to\pm\infty}(1-\frac{4}{x^2})\right]}$

$= 0$, so H.A. @ $y=0$

Intercepts: x-int: $(-3, 0)$

y-int: $(0, -3/4)$

Behaviour @ Asymptotes:

As x→-∞, f(x)→0 from below

As x→∞, f(x)→0 from above.

As x→2⁻, f(x)→-∞ As x→-2⁻, f(x)→∞

As x→2⁺, f(x)→∞ As x→-2⁺, f(x)→-∞

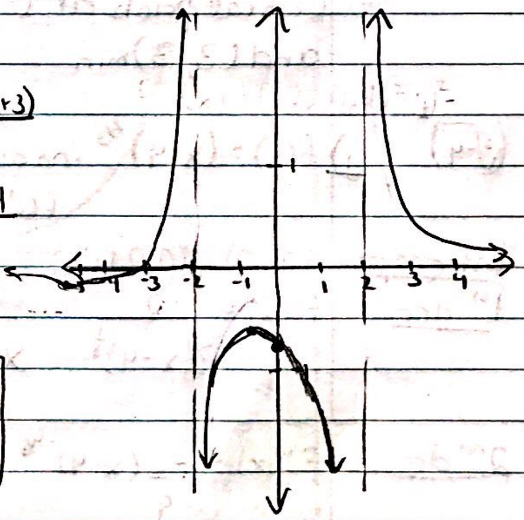
$f'(x) = \frac{x^2-4-2x(x+3)}{(x^2-4)^2}$

$= \frac{-x^2-6x-4}{(x^2-4)^2}$

$\therefore x^2+6x+4=0$

$x = \frac{-6\pm2\sqrt{5}}{2}$

$\boxed{\text{C.P.s at } (\overset{\sim -0.76}{-3+\sqrt{5}}, \overset{\sim -0.65}{\frac{\sqrt{5}}{10-6\sqrt{5}}}) \text{ and } (\overset{\sim -5.2}{-3-\sqrt{5}}, \underset{\sim -0.1}{\frac{-\sqrt{5}}{10+6\sqrt{5}}})}$



$f''(x) = \frac{(-2x-6)(x^2-4)^{\cancel{2}} - (-x^2-6x-4)(4x\cancel{(x^2-4)})}{(x^2-4)^{\cancel{4}3}}$

$= \frac{-2x^3+8x-6x^2+24+4x^3+24x^2+16x}{(x^2-4)^3}$

$= \frac{2x^3+18x^2+24x+24}{(x^2-4)^3}$

$x^3+9x^2+12x+12=0$

Ä!!

i) $y = \dfrac{x^2-3x+6}{x-1}$

$$\begin{array}{r} x-2 \\ x-1\overline{)\,x^2-3x+6} \\ \underline{x^2-x} \\ -2x+6 \\ \underline{-2x+2} \\ 4 \end{array}$$

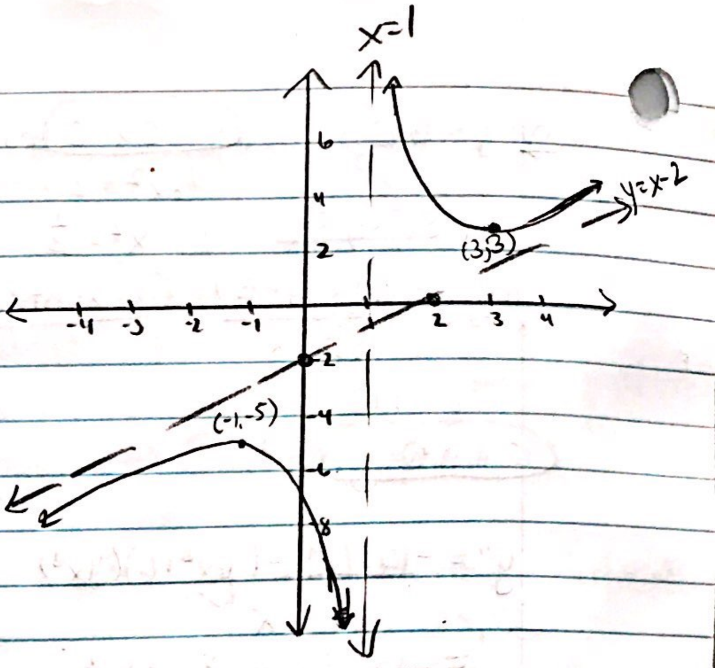
Asymptotes: V.A. @ $x=1$

As $x \to 1^-$, $y \to -\infty$

As $x \to 1^+$, $y \to \infty$

Oblique asymptote

$y = x-2.$



Intercepts: $(0,-6)$

no x-intercepts $(b^2-4ac \overset{!!}{=} -15)$

First derivative: $y' = \dfrac{(2x-3)(x-1)-x^2+3x-6}{(x-1)^2}$

$= \dfrac{x^2-2x-3}{(x-1)^2}$

$= \dfrac{(x-3)(x+1)}{(x-1)^2}$

Critical points at $(-1,-5)$ max and $(3,3)$ min

2nd derivative: $y'' = \dfrac{(2x-2)(x-1)^{\cancel{2}} - 2\cancel{(x-1)}(x+1)(x-3)}{(x-1)^{\cancel{4}3}}$

$= \dfrac{2x^2-4x+2-2x^2+4x+6}{(x-1)^3}$

$= \dfrac{8}{(x-1)^3}$ • no points of inflection.

When $x=-1$, $y''<0$, so CD ∩

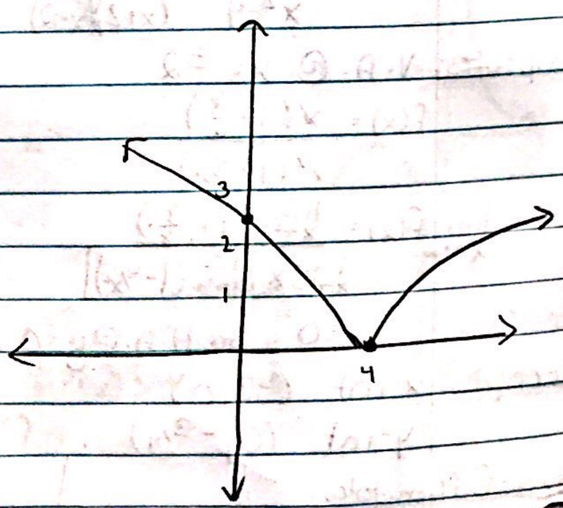
When $x=3$, $y''>0$, so CU ∪

$\left(\sqrt[3]{-4}\right)^2$ **j)** $f(x) = (x-4)^{2/3}$ • no asymptotes, but $f(x) \geq 0$ for all values of x.

Intercepts: $(4,0)$, $(0, 2.52)$

1st der: $f'(x) = \dfrac{2}{3(x-4)^{\frac{1}{3}}}$ $\quad \frac{2}{3}(x-4)^{-1/3}$

$x \neq 4$, so $(4,0)$ is a cusp.



2nd der: $f''(x) = -\frac{2}{9}(x-4)^{-4/3}$

$= \dfrac{-2}{9(x-4)^{+4/3}}$ $\quad x<4$, $f''(x) \leq 0$ ∩

$\quad x \geq 4$, $f''(x) < 0$ ∩

6. $y = ax^3 + bx^2 + cx + d$ max @ $(2,4)$, poi @ $(0,0)$

① $4 = 8a + 4b + 2c + d$ $y' = 3ax^2 + 2bx + c$ $y'' = 6ax + 2b$

④ $\boxed{0 = d}$ ② $0 = 12a + 4b + c$ ③ $\boxed{0 = 2b} \rightarrow b = 0$

$\therefore$ $8a + 2c = 4$, $12a + c = 0$ $\}$ $4a + c = 2$

⊖ $12a + c = 0$

$-8a = 2$

$\boxed{a = -1/4}$

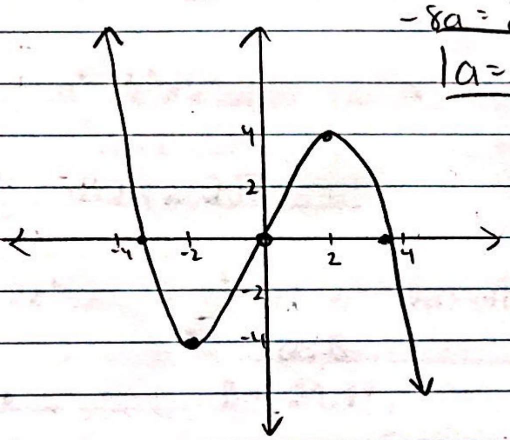
$12(-1/4) + c = 0$

$\boxed{c = 3}$

$$y = -\frac{1}{4}x^3 + 3x$$

As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to \infty$



Intercepts: $(0,0)$

$-\frac{1}{4}x(x^2 - 12) = 0$

$x = \pm 2\sqrt{3}$

$(\pm 2\sqrt{3}, 0)$

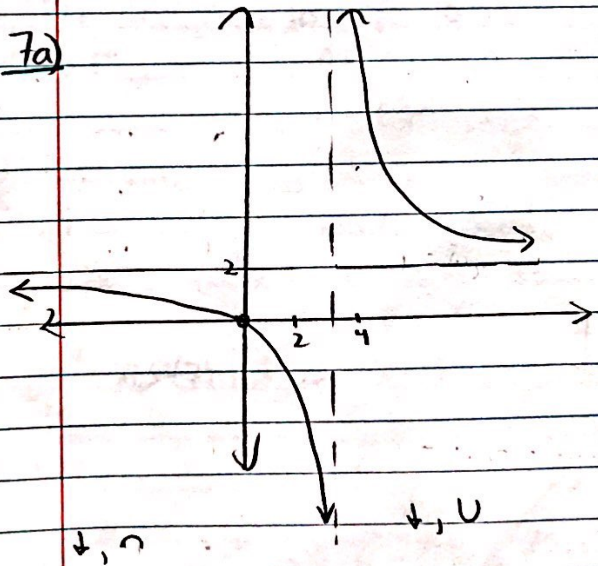
$y' = -\frac{3}{4}x^2 + 3$

$-3x^2 = -12$

$x = \pm 2$

CPs @ $(2,4)$, $(-2,-4)$

7a)



b)