4.10 Antiderivatives

1. $f(x) = 6x^2 - 8x + 3 \quad \Rightarrow \quad F(x) = 6\dfrac{x^{2+1}}{2+1} - 8\dfrac{x^{1+1}}{1+1} + 3x + C = 2x^3 - 4x^2 + 3x + C$

Check: $F'(x) = 2 \cdot 3x^2 - 4 \cdot 2x + 3 + 0 = 6x^2 - 8x + 3 = f(x)$

3. $f(x) = 1 - x^3 + 5x^5 - 3x^7 \quad \Rightarrow \quad F(x) = x - \dfrac{x^{3+1}}{3+1} + 5\dfrac{x^{5+1}}{5+1} - 3\dfrac{x^{7+1}}{7+1} + C = x - \frac{1}{4}x^4 + \frac{5}{6}x^6 - \frac{3}{8}x^8 + C$

5. $f(x) = 5x^{1/4} - 7x^{3/4} \quad \Rightarrow \quad F(x) = 5\dfrac{x^{1/4+1}}{\frac{1}{4}+1} - 7\dfrac{x^{3/4+1}}{\frac{3}{4}+1} + C = 5\dfrac{x^{5/4}}{5/4} - 7\dfrac{x^{7/4}}{7/4} + C = 4x^{5/4} - 4x^{7/4} + C$

7. $f(x) = \sqrt{x} + \sqrt[3]{x} = x^{1/2} + x^{1/3} \quad \Rightarrow \quad F(x) = \frac{1}{3/2}x^{3/2} + \frac{1}{4/3}x^{4/3} + C = \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$

9. $f(x) = \dfrac{10}{x^9} = 10x^{-9}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \dfrac{10x^{-8}}{-8} + C_1 = -\dfrac{5}{4x^8} + C_1 & \text{if } x < 0 \\ -\dfrac{5}{4x^8} + C_2 & \text{if } x > 0 \end{cases}$

See Example 1(c) for a similar exercise.

11. $g(t) = \dfrac{t^3 + 2t^2}{\sqrt{t}} = t^{5/2} + 2t^{3/2} \quad \Rightarrow \quad G(t) = \dfrac{t^{7/2}}{7/2} + \dfrac{2t^{5/2}}{5/2} + C = \frac{2}{7}t^{7/2} + \frac{4}{5}t^{5/2} + C$

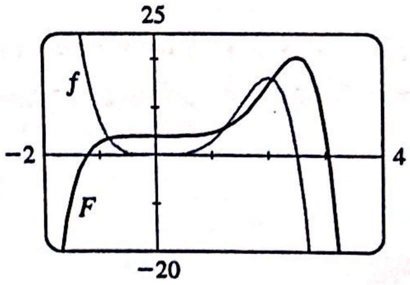
Note that g has domain $(0, \infty)$.

13. $f(t) = 3\cos t - 4\sin t \quad \Rightarrow \quad F(t) = 3(\sin t) - 4(-\cos t) + C = 3\sin t + 4\cos t + C$

15. $f(x) = 2x + \dfrac{5}{\sqrt{1 - x^2}} \quad \Rightarrow \quad F(x) = x^2 + 5\sin^{-1} x + C$

17. $f(x) = 5x^4 - 2x^5 \quad \Rightarrow \quad F(x) = x^5 - \frac{1}{3}x^6 + C.\ F(0) = 4 \quad \Rightarrow$

$C = 4$, so $F(x) = x^5 - \frac{1}{3}x^6 + 4$. The graph confirms our answer since $f(x) = 0$ when F has a local maximum, f is positive when F is increasing, and f is negative when F is decreasing.



19. $f''(x) = 6x + 12x^2 \quad \Rightarrow \quad f'(x) = 3x^2 + 4x^3 + C \quad \Rightarrow \quad f(x) = x^3 + x^4 + Cx + D$

21. $f''(x) = 1 + x^{4/5} \Rightarrow f'(x) = x + \frac{5}{9}x^{9/5} + C \Rightarrow$
$f(x) = \frac{1}{2}x^2 + \frac{5}{9} \cdot \frac{5}{14}x^{14/5} + Cx + D = \frac{1}{2}x^2 + \frac{25}{126}x^{14/5} + Cx + D$

23. $f'''(t) = e^t \Rightarrow f''(t) = e^t + C \Rightarrow f'(t) = e^t + Ct + D \Rightarrow f(t) = e^t + \frac{1}{2}Ct^2 + Dt + E$

25. $f'(x) = 1 - 6x \Rightarrow f(x) = x - 3x^2 + C$. $f(0) = C$ and $f(0) = 8 \Rightarrow C = 8$, so $f(x) = x - 3x^2 + 8$.

27. $f'(x) = 3\sqrt{x} - 1/\sqrt{x} = 3x^{1/2} - x^{-1/2} \Rightarrow f(x) = 3\left(\frac{1}{3/2}\right)x^{3/2} - \frac{1}{1/2}x^{1/2} + C \Rightarrow$
$2 = f(1) = 2 - 2 + C = C \Rightarrow f(x) = 2x^{3/2} - 2x^{1/2} + 2$

29. $f'(x) = 3\cos x + 5\sin x \Rightarrow f(x) = 3\sin x - 5\cos x + C \Rightarrow 4 = f(0) = -5 + C \Rightarrow C = 9 \Rightarrow$
$f(x) = 3\sin x - 5\cos x + 9$

31. $f'(x) = 2/x \Rightarrow f(x) = 2\ln|x| + C = 2\ln(-x) + C$ (since $x < 0$). Now
$f(-1) = 2\ln 1 + C = 2(0) + C = 7 \Rightarrow C = 7$. Therefore, $f(x) = 2\ln(-x) + 7, x < 0$.

33. $f''(x) = x \Rightarrow f'(x) = \frac{1}{2}x^2 + C \Rightarrow 2 = f'(0) = C \Rightarrow f'(x) = \frac{1}{2}x^2 + 2 \Rightarrow$
$f(x) = \frac{1}{6}x^3 + 2x + D \Rightarrow -3 = f(0) = D \Rightarrow f(x) = \frac{1}{6}x^3 + 2x - 3$

35. $f''(x) = x^2 + 3\cos x \Rightarrow f'(x) = \frac{1}{3}x^3 + 3\sin x + C \Rightarrow 3 = f'(0) = C \Rightarrow f'(x) = \frac{1}{3}x^3 + 3\sin x + 3$
$\Rightarrow f(x) = \frac{1}{12}x^4 - 3\cos x + 3x + D \Rightarrow 2 = f(0) = -3 + D \Rightarrow D = 5 \Rightarrow$
$f(x) = \frac{1}{12}x^4 - 3\cos x + 3x + 5$

37. $f''(x) = 6x + 6 \Rightarrow f'(x) = 3x^2 + 6x + C \Rightarrow f(x) = x^3 + 3x^2 + Cx + D$. $4 = f(0) = D$ and
$3 = f(1) = 1 + 3 + C + D = 4 + C + 4 \Rightarrow C = -5$, so $f(x) = x^3 + 3x^2 - 5x + 4$.

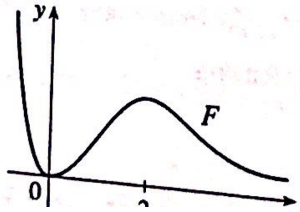
39. $f''(x) = x^{-3} \Rightarrow f'(x) = -\frac{1}{2}x^{-2} + C \Rightarrow f(x) = \frac{1}{2}x^{-1} + Cx + D \Rightarrow 0 = f(1) = \frac{1}{2} + C + D$ and
$0 = f(2) = \frac{1}{4} + 2C + D$. Solving these equations, we get $C = \frac{1}{4}$, $D = -\frac{3}{4}$, so $f(x) = 1/(2x) + \frac{1}{4}x - \frac{3}{4}$.

41. $f''(x) = x^{-2}, x > 0 \Rightarrow f'(x) = -1/x + C \Rightarrow f(x) = -\ln|x| + Cx + D = -\ln x + Cx + D$ (since
$x > 0$). $0 = f(1) = C + D$ and $0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C$ (since $D = -C$) $= -\ln 2 + C$
$\Rightarrow C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

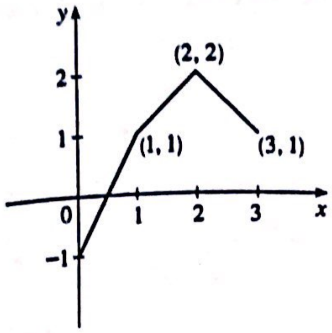
43. Given $f'(x) = 2x + 1$, we have $f(x) = x^2 + x + C$. Since f passes through $(1, 6)$, $6 = f(1) = 1^2 + 1 + C \Rightarrow$
$C = 4$. Therefore, $f(x) = x^2 + x + 4$ and $f(2) = 2^2 + 2 + 4 = 10$.

45. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

47. The graph of F will have a minimum at 0 and a maximum at 2, since $f = F'$ goes from negative to positive at $x = 0$, and from positive to negative at $x = 2$.



49.



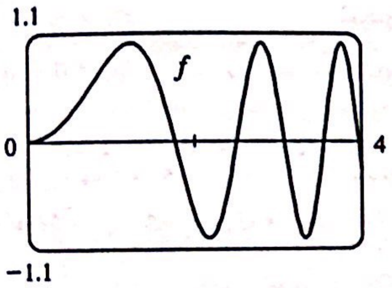
$$f'(x)=\begin{cases}2 & \text{if } 0\le x<1\\ 1 & \text{if } 1<x<2\\ -1 & \text{if } 2<x\le 3\end{cases}\quad\Rightarrow\quad f(x)=\begin{cases}2x+C & \text{if } 0\le x<1\\ x+D & \text{if } 1<x<2\\ -x+E & \text{if } 2<x\le 3\end{cases}$$

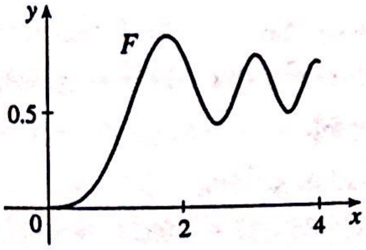
$f(0)=-1 \;\Rightarrow\; 2(0)+C=-1 \;\Rightarrow\; C=-1$. Starting at the point $(0,-1)$ and moving to the right on a line with slope 2 gets us to the point $(1,1)$. The slope for $1<x<2$ is 1, so we get to the point $(2,2)$. The line connecting $(1,1)$ to $(2,2)$ is $y=x$, so $D=0$. The slope for $2<x\le 3$ is -1, so we get to $(3,1)$. $f(3)=1 \;\Rightarrow\; -3+E=1 \;\Rightarrow\; E=4$.

$$\text{Thus, } f(x)=\begin{cases}2x-1 & \text{if } 0\le x<1\\ x & \text{if } 1<x<2\\ -x+4 & \text{if } 2<x\le 3\end{cases}$$

Note that f is continuous, but $f'(x)$ does not exist at $x=1$ or at $x=2$.

51.





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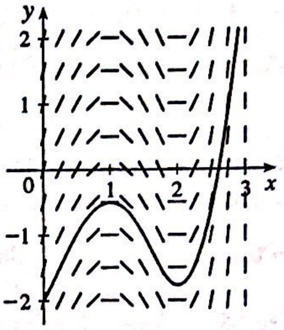
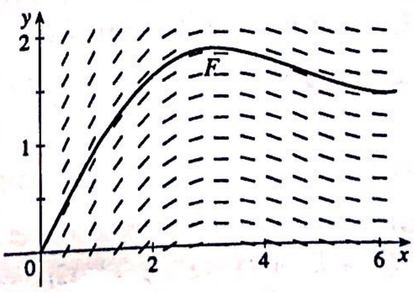


55.

x	$f(x)$
0	1
0.5	0.959
1.0	0.841
1.5	0.665
2.0	0.455
2.5	0.239
3.0	0.047

x	$f(x)$
3.5	−0.100
4.0	−0.189
4.5	−0.217
5.0	−0.192
5.5	−0.128
6.0	−0.047

We compute slopes (values of f) as in the table and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0,0)$.



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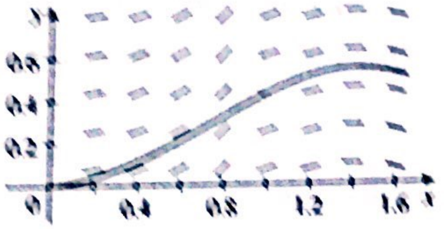

Remember that the values of f are the slopes of F at any x. For example, at $x = 1.4$, the slope of F is $f(1.4) = 0$.

59. $v(t) = s'(t) = \sin t - \cos t \Rightarrow s(t) = -\cos t - \sin t + C$. $s(0) = -1 + C$ and $s(0) = 0 \Rightarrow C = 1$, $s(t) = -\cos t - \sin t + 1$.

61. $a(t) = v'(t) = t - 2 \Rightarrow v(t) = \frac{1}{2}t^2 - 2t + C$. $v(0) = C$ and $v(0) = 3 \Rightarrow C = 3$, so $v(t) = \frac{1}{2}t^2 -$ and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + D$. $s(0) = D$ and $s(0) = 1 \Rightarrow D = 1$, and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + 1$.

63. $a(t) = v'(t) = 10\sin t + 3\cos t \Rightarrow v(t) = -10\cos t + 3\sin t + C \Rightarrow$ $s(t) = -10\sin t - 3\cos t + Ct + D$. $s(0) = -3 + D = 0$ and $s(2\pi) = -3 + 2\pi C + D = 12 \Rightarrow D =$ $C = \frac{6}{\pi}$. Thus, $s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3$.

65. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.
$v'(t) = a(t) = -9.8 \Rightarrow v(t) = -9.8t + C$, but $C = v(0) = 0$, so $v(t) = -9.8t \Rightarrow$
$s(t) = -4.9t^2 + D \Rightarrow D = s(0) = 450 \Rightarrow s(t) = 450 - 4.9t^2$.

(b) It reaches the ground when $0 = s(t) = 450 - 4.9t^2 \Rightarrow t^2 = 450/4.9 \Rightarrow t_1 = \sqrt{450/4.9} \approx 9.58$ s.

(c) $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. $v(t) = -9.8t + C \Rightarrow -5 = 0 + C \Rightarrow$
$v(t) = -9.8t - 5$. $s(t) = -4.9t^2 - 5t + D \Rightarrow 450 = s(0) = D \Rightarrow s(t) = -4.9t^2 - 5t + 450$.
$s(t) = 0 \Rightarrow t = \left(5 \pm \sqrt{8845}\right) / (-9.8) \Rightarrow t_1 \approx 9.09$ s.

67. By Exercise 66, $s(t) = -4.9t^2 + v_0 t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So
$[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6 v_0 t + v_0^2 = v_0^2 - 19.6\left(v_0 t - 4.9t^2\right)$. But $-4.9t^2 + v_0 t$ is just $s(t$
without the s_0 term, that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$.

69. Marginal cost $= 1.92 - 0.002x = C'(x) \Rightarrow C(x) = 1.92x - 0.001x^2 + K$. But
$C(1) = 1.92 - 0.001 + K = 562 \Rightarrow K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 =$
$C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

71. Taking the upward direction to be positive we have that for $0 \le t \le 10$ (using the subscript 1 to refer to $0 \le t \le 10$), $a_1(t) = -(9 - 0.9t) = v_1'(t) \Rightarrow v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \Rightarrow$
$v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \Rightarrow s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \Rightarrow$
$s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $s_1(10) = 100$, so it takes more than 10 seconds for the raindrop to fall.
for $t > 10$, $a(t) = 0 = v'(t) \Rightarrow v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \Rightarrow$
$v(t) = -55$. At 55 ft/s, it will take $100/55 \approx 1.8$ s to fall the last 100 ft. Hence, the total time is 11.8 s.

73. $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity is
50 mi/h$= 50 \cdot \frac{5280}{3600} = \frac{220}{3}$ ft/s. So $v(t) = kt + C$ and $v(0) = 44 \Rightarrow C = 44$. Thus, $v(t) = kt + 44 \Rightarrow$
$\frac{220}{3} = v(5) = 5k + 44 \Rightarrow k = \frac{88}{15} \approx 5.87$ ft/s^2.

75. Using Exercise 66 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t
$s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t \Rightarrow -32t = -120 \Rightarrow t = 3.75$, so
$0 = s(3.75) = -16(3.75)^2 + h \Rightarrow h = 16(3.75)^2 = 225$ ft.