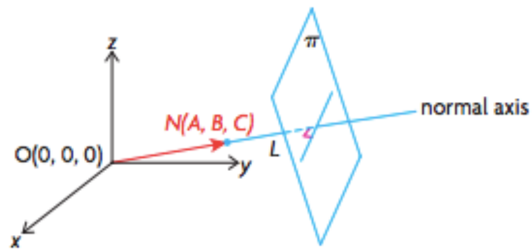
Date: ____________________

8.5 The Cartesian Equation of a Plane

Because a plane is two dimensional, we can find a scalar (Cartesian) equation for it. This is similar to the Cartesian equation of a line ($Ax + By + C = 0$) in R^2.

Every plane has one unique line that is perpendicular to it and passes through the origin (normal axis).

To find the Cartesian equation of a plane, we need to know the direction vector for the normal axis (normal vector) and any point on the plane.



Cartesian Equation of a Plane:

$Ax + By + Cz + D = 0$

x, y, z A, B, C

Example: The point A(1, -2, 5) is a point on the plane with normal $\vec{n} = (5, -1, 4)$. Determine the Cartesian equation of the plane.

Method 1: Substitution

$5(1) - 1(-2) + 4(5) + D = 0$

$5 + 2 + 20 + D = 0$

$D = -27$

$\boxed{5x - y + 4z - 27 = 0}$

Method 2: The Dot Product

$P(x, y, z)$

$\vec{AP} = (x-1, y+2, z-5)$

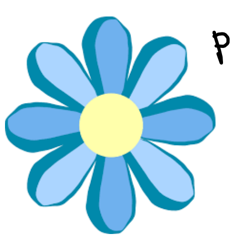
Direction vector for plane

$\vec{AP} \cdot \vec{n} = 0$

$5(x-1) - 1(y+2) + 4(z-5) = 0$

$5x - 5 - y - 2 + 4z - 20 = 0$

$\boxed{5x - y + 4z - 27 = 0}$



When the normal vector is NOT given, we have to be able to find it. This means that we need to find the CROSS PRODUCT of two direction vectors that are on the plane.

Example: Determine the Cartesian equation of the plane containing the points A(-2, 1, 5), B(2, 3, 4) and C(-1, 4, 6).

$\vec{AB} = (4, 2, -1)$

$\vec{AC} = (1, 3, 1)$

$$\begin{array}{cc} 2 & 3 \\ -1 & 1 \\ 4 & 1 \\ 2 & 3 \end{array}$$

$\vec{n} = \vec{AB} \times \vec{AC}$

$= (2+3, -1-4, 12-2)$

$\vec{n} = (5, -5, 10)$

or $\vec{n} = (1, -1, 2)$

$Ax + By + Cz + D = 0$

$(1)(-2) + (-1)(1) + 2(5) + D = 0$

$-2 - 1 + 10 + D = 0$

$D = -7$

$\boxed{x - y + 2z - 7 = 0}$

Example: Determine the Cartesian equation of a plane whose vector equation is r = (1, 2, -1) + s(1, 0, 2) + t(-1, 3, 4).

$$\begin{array}{cc} 0 & 3 \\ 2 & 4 \\ 1 & -1 \\ 0 & 3 \end{array}$$

$\vec{n} = (0-6, -2-4, 3-0)$

$= (-6, -6, 3)$

or $\vec{n} = (2, 2, -1)$

$Ax + By + Cz + D = 0$

$2(1) + 2(2) - 1(-1) + D = 0$

$2 + 4 + 1 + D = 0$

$D = -7$

$2x + 2y - z - 7 = 0$

If we need to find the other forms of an equation given a Cartesian equation, things get a little scarier because you have to assign parameters (s and t) for variables. You will choose a variable to call s and a variable to call t. Then you can express the remaining variable in terms of s and t, and you have created your parametric equations!

Example: Determine the parametric and vector equations for a plane represented by $-2x + 5y - z + 4 = 0$.

$\vec{n} = (-2, 5, -1)$

Let $y = s$, $z = t$

$-2x + 5s - t + 4 = 0$

$-2x = -5s + t - 4$

Parametric
$$\begin{cases} x = \frac{5}{2}s - \frac{1}{2}t + 2 \\ y = s \\ z = t \end{cases}$$

$\vec{r} = (2, 0, 0) + s\left(\frac{5}{2}, 1, 0\right) + t\left(-\frac{1}{2}, 0, 1\right)$

Parallel and Perpendicular Planes

If two planes are parallel, how are their normal vectors related?

Also parallel

If two planes are perpendicular, how are their normal vectors related?

Perpendicular to each other

What do we mean if we say that two planes are coincident? How would we show that two planes are parallel but not coincident?

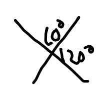
Coincident → identical

same normal/dir. vectors, diff. pt. not on the other plane.

How can we find the angle between intersecting planes?

$\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

Example: Show that the planes represented by $x - y - 2z + 3 = 0$ and $2x + y - z + 2 = 0$ are not perpendicular, and then find the angle between them.



$\vec{n}_1 \cdot \vec{n}_2 = (1, -1, -2) \cdot (2, 1, -1)$

$= 2 - 1 + 2$

$= 3$

$\vec{n}_1 \cdot \vec{n}_2 \neq 0$, so not $\perp$

$\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

$= \dfrac{3}{(\sqrt{6})(\sqrt{6})}$

$= \frac{1}{2}$

$\theta = 60°$