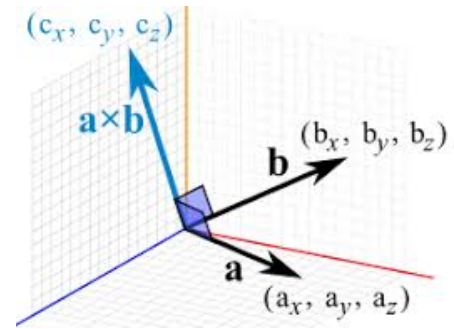


Date:

7.6 The Cross Product of Two Vectors

The cross product of two vectors, \vec{a} and \vec{b} , in \mathbb{R}^3 is defined as the vector that is perpendicular to both a and b . This is also referred to as the vector product because it **results in a vector quantity**, unlike the scalar dot product.



Formula for the Magnitude of the Cross Product of Geometric Vectors

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin \theta$ (often used in physics when you know the angle between two vectors.)

Formula for the Cross Product of Algebraic Vectors (More important!!)

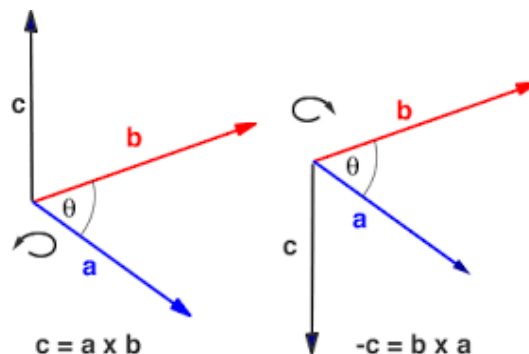
$k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ is a vector perpendicular to both a and b , when $k \in \mathbb{R}$.

If $k = 1$, then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

If $k = -1$, then $\vec{b} \times \vec{a} = (b_2a_3 - b_3a_2, b_3a_1 - b_1a_3, b_1a_2 - b_2a_1)$

There are an infinite number of vectors that are perpendicular to a and b , but they are all scalar multiples of one another. The simplest values of k to work with are 1 and -1, so our formula is based upon this assumption. A detailed development of this formula is presented on p. 403 on your text book.

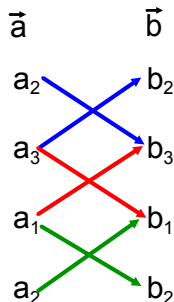
The cross product is NOT commutative! $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$!



The formula for calculating the cross product of algebraic vectors is not easy to remember. Use the following trick so that you don't have to memorize the formula!

For $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$,

- 1) List your components in the order shown below.



- 2) Multiply in the direction of the arrows.

$$x = a_2b_3 - a_3b_2$$

$$y = a_3b_1 - a_1b_3$$

$$z = a_1b_2 - a_2b_1$$

$$3) \vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

Example: For vectors $\vec{a} = (-2, 3, 6)$ and $\vec{b} = (4, 1, 5)$, determine $\vec{a} \times \vec{b}$ and

| | | |
|---------------|--------------|----------------------------------------------------------|
| \vec{a} | \vec{b} | $\vec{a} \times \vec{b} = (15 - 24, 24 + 10, -2 - 12)$ |
| \vec{a} | \vec{b} | |
| 3 | 1 | $\vec{a} \times \vec{b} = (9, 34, -14)$ |
| 6 | 5 | |
| -2 | 4 | $ \vec{a} \times \vec{b} = \sqrt{9^2 + 34^2 + (-14)^2}$ |
| 3 | 1 | |
| | | $= \sqrt{1455}$ |

Use the dot product to check your answer. What are you expecting?

Check w/ \vec{a}

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = -18 + 102 - 84 = 0 \quad \checkmark$$

Check w/ \vec{b}

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = 36 + 34 - 70 = 0 \quad \checkmark$$

$\vec{a} \cdot \vec{b} = 0$ if $\vec{a} \perp \vec{b}$

Properties of the Cross Product

Let \vec{p} , \vec{q} , and \vec{r} be three vectors in R^3 , and let $k \in \mathbf{R}$.

Vector multiplication is not commutative: $\vec{p} \times \vec{q} = -(\vec{q} \times \vec{p})$,

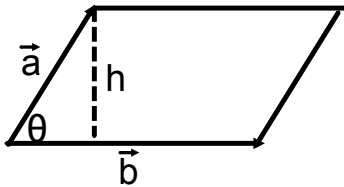
Distributive law for vector multiplication: $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$,

Scalar law for vector multiplication: $k(\vec{p} \times \vec{q}) = (k\vec{p}) \times \vec{q} = \vec{p} \times (k\vec{q})$,

Area of a Parallelogram - A Geometric Application of the Cross Product

For a parallelogram with sides \vec{a} and \vec{b} , the area is $|\vec{a} \times \vec{b}|$.

Proof:



$$\sin \theta = \frac{h}{|\vec{a}|} \quad b = |\vec{b}|$$

$$A = bh$$

$$A = |\vec{b}| |\vec{a}| \sin \theta$$

$$\therefore |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

← cross product formula for geo. vector.

Example: Find the area of the parallelogram formed by $\vec{a} = (1, 2, 4)$ and $\vec{b} = (3, 1, 4)$.

$$(|\vec{a} \times \vec{b}|)$$

$$\begin{array}{r} \vec{a} \\ \hline 1 \\ 2 \\ 4 \end{array} \quad \begin{array}{r} \vec{b} \\ \hline 3 \\ 1 \\ 4 \end{array}$$

$$\begin{aligned} \vec{a} \times \vec{b} &= (8 - 4, 12 - 4, 1 - 6) \\ &= (4, 8, -5) \end{aligned}$$

$$|\vec{a} \times \vec{b}| = A$$

$$= \sqrt{4^2 + 8^2 + (-5)^2}$$

$$= \sqrt{105} \text{ sq. units}$$

