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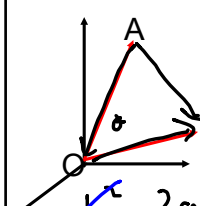
### 7.4 The Dot Product of Algebraic Vectors

Remember that algebraic vectors are those that can be expressed in component form. Dot product calculations are actually easier for algebraic vectors.

Calculating the Dot Product for Algebraic Vectors in  $\mathbb{R}^3$ :

$$\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos \theta = x_1 y_1 + x_2 y_2 + x_3 y_3, \text{ where } \vec{x} = (x_1, x_2, x_3) \text{ and } \vec{y} = (y_1, y_2, y_3)$$

To show why this formula is true, find the dot product of the nonzero vectors  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$  and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ .



$|\vec{a}-\vec{b}|^2 = (a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2$   
 $\vec{a}-\vec{b} = (b_1-a_1, b_2-a_2, b_3-a_3)$   
 $(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\vec{a} \cdot \vec{b}$   
 $(-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\vec{a} \cdot \vec{b}) \div -2$   
 $a_1b_1 + a_2b_2 + a_3b_3 = \vec{a} \cdot \vec{b}$

Now rearrange the formula to determine how to find the angle between two vectors joined tail to tail.

$$\cos \theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|\vec{a}| |\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

Example: Given the vectors  $\vec{a} = (-1, 2, 4)$  and  $\vec{b} = (3, -2, 1)$ , calculate the angle between them.

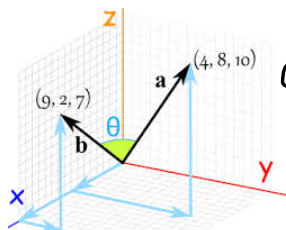
① Find  $\vec{a} \cdot \vec{b}$ .

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (-1, 2, 4) \cdot (3, -2, 1) \\ &= (-1)(3) + (2)(-2) + 4(1) \\ &= -3 \end{aligned}$$

② Find  $\theta$

$$\cos \theta = \frac{-3}{(\sqrt{17})(\sqrt{14})}$$

$$\theta = 100.1^\circ$$



③ Find  $|\vec{a}|$  and  $|\vec{b}|$

$$\begin{aligned} |\vec{a}| &= \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{17} \\ |\vec{b}| &= \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14} \end{aligned}$$

Practice Problems:

- 1) For what values of  $m$  are the vectors  $\vec{p} = (m, m, -3)$  and  $\vec{q} = (m, -3, 6)$  perpendicular?

$$\hookrightarrow \vec{p} \cdot \vec{q} = 0$$

$$(m, m, -3) \cdot (m, -3, 6) = 0$$

$$m^2 - 3m - 18 = 0$$

$$(m-6)(m+3) = 0$$

$$\therefore m = 6, m = -3$$

- 2) Find a vector perpendicular to both  $\vec{a} = (1, 5, -1)$  and  $\vec{b} = (-3, 1, 2)$

$$\vec{n} = (x, y, z) \leftarrow (x, y, 16y)$$

2 eqns, 3 unknowns

$$\textcircled{1} \vec{a} \cdot \vec{n} = 0$$

$$\textcircled{2} \vec{b} \cdot \vec{n} = 0$$

$$x + 5y - z = 0$$

$$-3x + y + 2z = 0$$

$$x = -5y + z \quad \longrightarrow \quad -3(-5y + z) + y + 2z = 0$$

$$15y - 3z + y + 2z = 0$$

$$x = -5y + 16y$$

$$x = 11y \quad \therefore \vec{n} = (11y, y, 16y) \quad 16y - z = 0$$

$$\textcircled{16y} = z$$

- 3) Find  $\vec{u} \cdot \vec{v}$ , where  $\vec{u} = 3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$  and  $\vec{v} = -7\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ .

$$\vec{u} = (3, 2, -7) \quad \vec{v} = (-7, 4, -3)$$

$$\vec{u} \cdot \vec{v} = (3)(-7) + (2)(4) + (-7)(-3)$$

$$= -21 + 8 + 21$$

$$= 8$$

