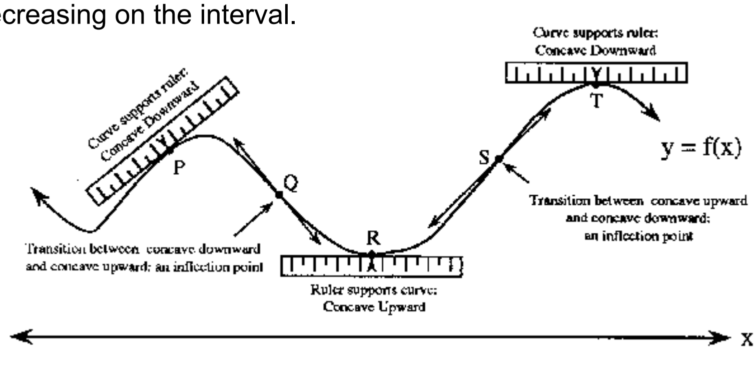
Date: ____________________

4.4 Concavity and Points of Inflection

Concavity refers to the rate of change of the tangent slope. A graph is **concave up** on an interval if the tangent slopes are increasing on that interval. A graph is **concave down** on an interval if the tangent slopes are decreasing on the interval.



Points of Inflection are the points on the graph where concavity changes. Sometimes they are obvious (like in a cubic function), and sometimes they are not (see the diagram above).

When we are dealing with concavity, we are concerned with finding points of inflection, as well as with identifying the intervals on which a function is concave up and concave down.

How can we find the rate of change of the tangent slope?

Second derivative

What are we looking for to decide if the function is concave up or concave down for a given value of x?

If second derivative is ⊕, concave up (☺)

If " " " ⊖, concave down (☹)

How can this help us to classify critical points as maxima, minima, or neither?

Concave up → min

Concave down → max

$f''(c) = 0$

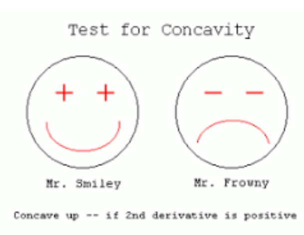
Point of inflection

use 1st order test.

*This is called the second derivative test, and is usually more efficient than the first derivative test to classify critical points!

How can we find the coordinates of the points of inflection for a function?

Set the second derivative equal to zero



Example 1:

Determine any critical points and points of inflection for the graph of $f(x) = x^3 + 2x^2 - 4x + 3$. Use the second derivative test to classify the critical points, and state the intervals on which the graph is concave up and concave down. Sketch the graph.

$f'(x) = 3x^2 + 4x - 4$

$(3x-2)(x+2) = 0$

$x = \frac{2}{3} \quad x = -2$ ← critical values

Critical Points:
$f(\frac{2}{3}) = \frac{41}{27}$ ← min
$f(-2) = 11$ ← max

C.D. $x \in (-\infty, -\frac{2}{3})$

C.U. $x \in (-\frac{2}{3}, \infty)$

$f''(x) = 6x + 4$

$6x + 4 = 0$ ← point of inflection

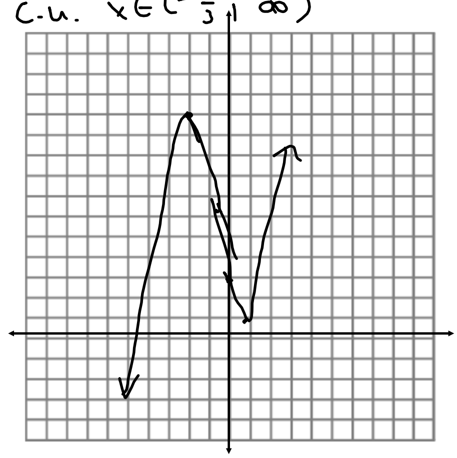
$x = -\frac{2}{3}$

$f(-\frac{2}{3}) = \frac{169}{27}$

Second der. test

$f''(\frac{2}{3}) = 8$ (+) → C.U. ∪

$f''(-2) = -8$ (−) → C.D. ∩



Example 2:

Sketch the graph of the function $g(x) = x^{\frac{1}{3}}$.

Intercept: $(0,0)$

$g'(x) = \frac{1}{3}x^{-2/3}$

$= \frac{1}{3(\sqrt[3]{x})^2}$ can't be zero

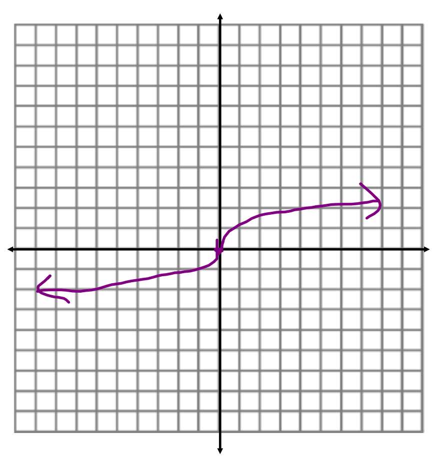
Critical value is zero.

g'(x) is undefined where g(x) is defined. (Tangent slope undefined)

$g''(x) = -\frac{2}{9}x^{-5/3}$

Still undefined at $x = 0$

Interval	$x < 0$	$x = 0$	$x > 0$
g"(x)	+	undef	−
g(x)	∪	∫	∩



The use of the second derivative allows us to refine our approach to curve sketching, as we can now use the second derivative to classify critical points and identify points of inflection.

Example 3:

Use the appropriate steps for curve sketching to accurately sketch the graph of $f(x) = \dfrac{4x^2 - 3}{x^3}$.

Use f(x) to identify asymptotes and intercepts.

V.A: $x = 0$

x-ints: $4x^2 - 3 = 0$
$4x^2 = 3$
$x = \pm\frac{\sqrt{3}}{2}$

H.A: $f(x) = \dfrac{4x^{\cancel{2}}(1 - \frac{3}{4x^2})}{x^{\cancel{3}}(1)}$

$\lim_{x\to\infty} f(x) = \frac{4}{x} \leftarrow 0$
$= 0$ $\boxed{f(x) = 0}$

Find f'(x) and f''(x) to identify critical points, intervals of increase, decrease, and concavity.

$f'(x) = \dfrac{8x(x^3) - (4x^2-3)(3x^2)}{x^6}$

$= \dfrac{8x^4 - 12x^4 + 9}{x^4}$

$= \dfrac{-4x^2 + 9}{x^4}$

$f''(x) = \dfrac{-8x(x^4) - 4x^3(-4x^2+9)}{x^8}$

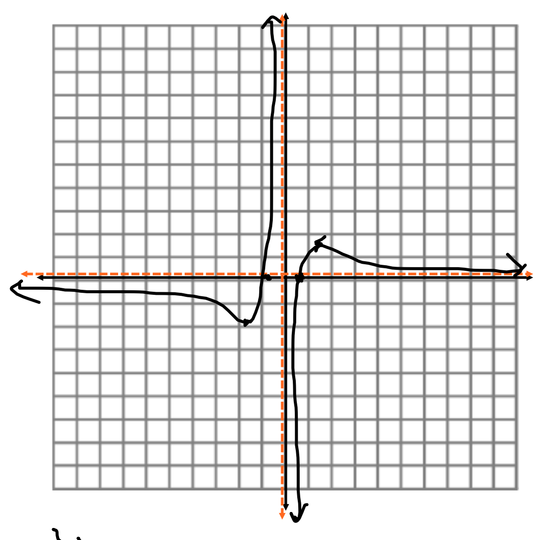
$= \dfrac{-8x^5 + 16x^5 - 36x^3}{x^8}$

$= \dfrac{+8x^2 - 36}{x^5}$

$8x^2 - 36 = 0$
$x^2 = \frac{9}{2}$
$x = \pm\frac{3}{\sqrt{2}}$ or $\frac{3\sqrt{2}}{2}$

Critical Points
$-4x^2 + 9 = 0$
$x^2 = \frac{9}{4}$
$x = \pm\frac{3}{2}$

$f(\frac{3}{2}) = \frac{16}{9}$ $f(-\frac{3}{2}) = -\frac{16}{9}$
max min



$f''(\frac{3}{2}) = -$ CD ∩

$f''(-\frac{3}{2}) = +$ CU ∪