Date: ____________________

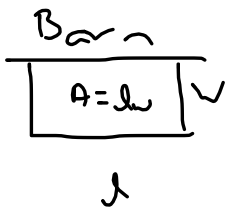
3.3 Optimization Problems

Whenever you are asked to find the maximum or minimum value in a situation, you are solving an optimization problem. We have seen this in previous courses, particularly with respect to quadratic relations.

Example:

A apple orchard has 1200 m of fencing to enclose their trees. They plan to use the side of a barn as one side of the enclosure. Determine the dimensions that they should use to maximize area.

Method #1 - Use Existing Knowledge



① $A = lw$

$= (1200 - 2w)(w)$

$= -2w^2 + 1200w$

② $2w + l = 1200\text{m}$

$l = 1200 - 2w$

Find vertex: $w = \frac{-1200}{-4}$

$= 300\text{m}$

∴ 300m × 600m will maximize the area.

Method #2 - Try Some Calculus!

$0 \le w \le 600$

$A(w) = -2w^2 + 1200w$

$A'(w) = -4w + 1200$

$4w = 1200$

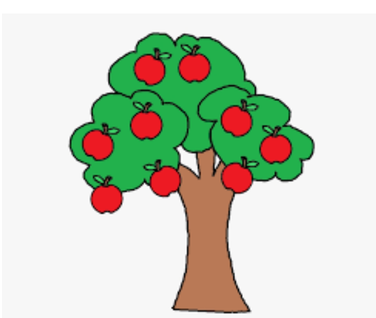
$w = 300$

$A(0) = 0$

$A(600) = 0$

max → $A(300) = 180\,000\text{m}^2$

∴ The dimensions are 300m × 600m.

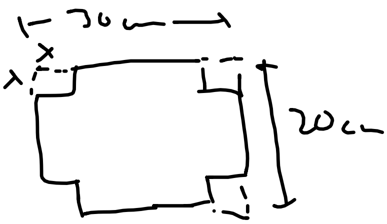


Tips to Approaching Optimization Problems:

- Identify what the problem is asking, as well as what it is offering in the way of information, and create a diagram.
- Draw on previous knowledge (area, perimeter, surface area, volume, Pythagorean theorem, similar triangles, distance, speed, time, etc.) to develop two equations in two unknowns.
- Identify constraints on those equations (given in the question or implied).
- Use substitution to create one equation, and then find its derivative so that you can find turning points. Check the end points of your domain in the original function as well so that you are sure that you found the optimal value.

Another Famliar Example:

An open-topped box can be created by cutting congruent squares from each of the four corners of a piece of cardboard that has dimensions of 20 cm by 30 cm and folding up the sides. Determine the dimensions of the square that will maximize the volume of the box.



Let x be the width of the square.

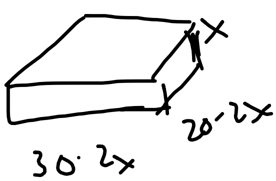
$V = lwh$

$V(x) = (30-2x)(20-2x)(x)$

$= 600x - 60x^2 - 40x^2 + 4x^3$

$= 4x^3 - 100x^2 + 600x$

$V'(x) = 12x^2 - 200x + 600$



$0 \le x \le 10$

$0 = 4(3x^2 - 50x + 150)$

$x = \frac{50 \pm \sqrt{700}}{6}$

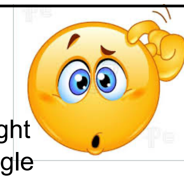
$\cancel{x = 12.74}$, $x = 3.92$

∴ The square is 3.92 cm x 3.92 cm.

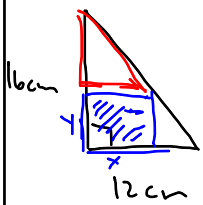
Bonus Question: How is this question different from the ones that we asked you in Advanced Functions?

We can find max/min now!

Two Less Familiar Examples:



1) Determine the area of the largest rectangle that can be inscribed inside a right triangle if the legs adjacent to the right triangle are 12 cm and 16 cm long. Two sides of the rectangle lie along the legs.



① $A = xy$

$A(y) = \left(\frac{48-3y}{4}\right)y$

$= -\frac{3}{4}y^2 + 12y$

$A'(y) = -\frac{3}{2}y + 12$

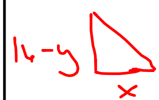
$\frac{3}{2}y = 12$

$3y = 24$

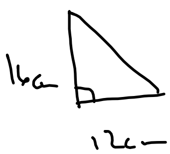
$y = 8\text{ cm}$

$x = \frac{48-24}{4}$

$x = 6\text{ cm}$



② Similar triangles



$\frac{16-y}{16} = \frac{x}{12}$

$\frac{192-12y}{16} = x$

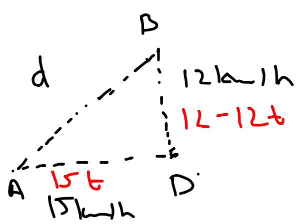
$x = \frac{48-3y}{4}$

$\therefore$ The max area is 48 cm^2.

2) A boat leaves a dock at 2:00 pm heading west at 15 km/h. Another boat heads south at 12 km/h and reaches the same dock at 3:00 pm. When were the boats closest to each other?

Minimize hypotenuse

Let t be time since A departed.



$0 \le t \le 1$

A: $d = 15t$

B: $d = 12 - 12t$

$c^2 = a^2 + b^2$

$[d(t)]^2 = (15t)^2 + (12-12t)^2$

$= 225t^2 + 144 - 288t + 144t^2$

$d(t) = \sqrt{369t^2 - 288t + 144}$

$= (369t^2 - 288t + 144)^{\frac{1}{2}}$

$d'(t) = \frac{1}{2}(369t^2 - 288t + 144)^{-\frac{1}{2}}(738t - 288)$

$0 = \frac{738t - 288}{2(369t^2 - 288t + 144)^{\frac{1}{2}}}$

$\frac{738t}{738} = \frac{288}{738}$

$t = 0.39$ hours.

0.39×60
$= 23.4$ min

$\therefore$ They are closest at 2:23 pm.