3.1 Higher-Order Derivatives, Velocity, and Acceleration

1. $v(t) = 2t - t^2$

$v(1) = 1\,m/s$

$v(5) = -15\,m/s$ ← faster, opposite direction

4a i) $t = 3$

ii) $1 < t < 3$

iii) $3 < t < 5$

b i) $t = 3,\ t = 7$

ii) $1 < t < 3,\ t > 7$

iii) $3 < t < 7$

2a) $y = x^{10} + 3x^6$

$y' = 10x^9 + 18x^5$

$y'' = 90x^8 + 90x^4$

$= 90x^4(x^4 + 1)$

b) $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$ or $\dfrac{1}{2}x^{-1/2}$

$f''(x) = -\dfrac{1}{4}x^{-3/2}$

$= \dfrac{-1}{4(\sqrt{x})^3}$

c) $y = (1-x)^2$

$y' = -2(1-x)$

$= 2x - 2$

$y'' = 2$

d) $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$

$h'(x) = 12x^3 - 12x^2 - 6x$

$h''(x) = 36x^2 - 24x - 6$

$= 6(6x^2 - 4x - 1)$

e) $y = 4x^{3/2} - x^2$

$y' = 6x^{1/2} - 2x$

$y'' = 3x^{-1/2} - 2$

$= \dfrac{3}{\sqrt{x}} - 2$

f) $f(x) = \dfrac{2x}{x+1}$

$f'(x) = \dfrac{2(x+1) - 2x}{(x+1)^2}$

$= \dfrac{2}{(x+1)^2}$

$f''(x) = \dfrac{-2(2(x+1))}{(x+1)^4}$

$= \dfrac{-4}{(x+1)^3}$

g) $y = x^2 + x^{-2}$

$y' = 2x - 2x^{-3}$

$y'' = 2 + 6x^{-4}$

$= 2 + \dfrac{6}{x^4}$

h) $g(x) = (3x-6)^{1/2}$

$g'(x) = \frac{1}{2}(3x-6)^{-1/2}(3)$

$= \frac{3}{2}(3x-6)^{-1/2}$

$g''(x) = -\frac{3}{4}(3x-6)^{-3/2}(3)$

$= -\dfrac{9}{4(3x-6)^{3/2}}$

i) $y = (2x+4)^3$

$y' = 3(2x+4)^2(2)$

$= 6(2x+4)^2$

$y'' = 12(2x+4)(2)$

$= 24(2x+4)$

$= 48x + 96$

j) $h(x) = x^{5/3}$

$h'(x) = \frac{5}{3}x^{2/3}$

$h''(x) = \frac{10}{9}x^{-1/3}$

$= \dfrac{10}{9\sqrt[3]{x}}$

3a) $s(t) = 5t^2 - 3t + 15$

$s'(t) = v(t) = 10t - 3$

$s''(t) = v'(t) = a(t) = 10$

b) $s(t) = 2t^3 + 36t - 10$

$v(t) = 6t^2 + 36$

$a(t) = 12t$

3c) $s(t) = t - 8 + 6t^{-1}$

$v(t) = 1 - 6t^{-2}$

$a(t) = 12t^{-3}$

$= \dfrac{12}{t^3}$

d) $s(t) = (t-3)^2$

$v(t) = 2(t-3)$

$a(t) = 2$

e) $s(t) = (t+1)^{1/2}$

$v(t) = \frac{1}{2}(t+1)^{-1/2}$

$a(t) = -\frac{1}{4}(t+1)^{-3/2}$

$= -\dfrac{1}{4(t+1)^{3/2}}$

f) $s(t) = \dfrac{9t}{t+3}$

$v(t) = \dfrac{9(t+3) - 9t}{(t+3)^2}$

$= \dfrac{27}{(t+3)^2}$

$a(t) = \dfrac{-54(t+3)}{(t+3)^4}$

$= \dfrac{-54}{(t+3)^3}$

5. $s = \frac{1}{3}t^3 - 2t^2 + 3t,\ t \geq 0$

a) $s' = t^2 - 4t + 3$ (v)

$s'' = 2t - 4$ (a)

b) When $v = 0$:

$(t-3)(t-1) = 0$, so 1s + 3s.

c) When $s = 0$:

$\frac{1}{3}t(t^2 - 6t + 9) = 0$

$\frac{1}{3}t(t-3)^2 = 0$, so $t = 3s$.

6a) $s(t) = -\frac{1}{3}t^2 + t + 4$

$s'(t) = -\frac{2}{3}t + 1$

$s'(1) = \frac{1}{3}$ (+)

$s'(4) = -\frac{5}{3}$ (−)

b) $s(t) = t^3 - 6t^2 + 9t$

$s'(t) = 3t^2 - 12t + 9$

$s'(1) = 0$ (stopped)

$s'(4) = 9$ (+)

c) $s(t) = t^3 - 7t^2 + 10t$

$s'(t) = 3t^2 - 14t + 10$

$s'(1) = -3$ (−)

$s'(4) = 2$ (+)

7. $s(t) = t^2 - 6t + 8$

a) $s'(t) = 2t - 6$

b) $2t - 6 = 0$

$t - 3 = 0$

$\boxed{t = 3}$

10. $s(t) = t^{5/2}(7 - t)$

$= 7t^{5/2} - t^{7/2}$

a) $v(t) = \frac{35}{2}t^{3/2} - \frac{7}{2}t^{5/2}$

$= \frac{7}{2}t^{3/2}(5 - t)$

$a(t) = \frac{105}{4}t^{1/2} - \frac{35}{4}t^{3/2}$

$= \frac{35}{4}t^{1/2}(3 - t)$

b) When does $v(t) = 0$?

$\frac{7}{2}t^{3/2}(5 - t) = 0$

$\boxed{t = 5s.}$

c) 5s.

d) When does $a(t) = 0$?

$\frac{35}{4}t^{1/2}(3 - t) = 0$

$t = 0, \; t = 3$

$a(t) > 0$ from 0 to 3s.

e) $t^{5/2}(7 - t) = 0$

$\boxed{t = 7s}$

11. $h(t) = -5t^2 + 25t$

a) $h'(t) = -10t + 25$

$h'(0) = 25$ m/s

b) $-10t + 25 = 0$

$-10t = -25$

$t = 2.5s$

$h(2.5) = -5(2.5)^2 + 25(2.5)$

$= 31.25$m

c) $-5t^2 + 25t = 0$

$-5t(t - 5) = 0$

$t = 0, \; \boxed{t = 5s}$

$h'(5) = -10(5) + 25$

$= -25$ m/s

8. $s(t) = 40t - 5t^2$

a) $s'(t) = 40 - 10t$

$40 - 10t = 0$

$t = 4$

∴ The object stops rising at 4s.

b) $s(4) = 40(4) - 5(16)$

$= 80$m

∴ The max ht. is 80m.

9. $s(t) = 8 - 7t + t^2$

a) $s'(t) = -7 + 2t$

$s'(5) = -7 + 10$

$= 3$ m/s

b) $s''(t) = 2$

$s''(5) = 2$ m/s²

13a) $s(t) = 10 + 6t - t^2$

$v(t) = 6 - 2t$

$a(t) = -2$

b) $s(t) = t^3 - 12t - 9$

$s'(t) = 3t^2 - 12$

$s''(t) = 6t$

12. $s(t) = 6t^2 + 2t$

a) $s'(t) = 12t + 2$

$s''(t) = 12.$

Crosses the finish line when $s(t) = 400$.

$6t^2 + 2t = 400$

$3t^2 + t - 200 = 0$

$(3t + 25)(t - 8) = 0$

$t = -\frac{25}{3}, \; t = 8$

$s'(8) = 98$ m/s

$s''(8) = 12$ m/s²

b) $6t^2 + 2t - 60 = 0$

$3t^2 + t - 30 = 0$

$(3t + 10)(t - 3) = 0$

$t = -\frac{10}{3}, \; t = 3$

$s'(3) = 38$ m/s

14. $s(t) = t^5 - 10t^2$

$s'(t) = 5t^4 - 20t$

$s''(t) = 20t^3 - 20$

$20(t^3 - 1) = 0$

$20(t - 1)(t^2 + t + 1)$

15. $s(t) = kt^2 + (6k^2 - 10k)t + 2k$

a) $v(t) = 2kt + 6k^2 - 10k$

$= 2k(t + 3k - 5)$

$a(t) = 2k$ ← constant

b) $t = 5 - 3k$

$s(5 - 3k) = k(5 - 3k)^2 + ((6k^2 - 10k)(5 - 3k)) + 2k$

3.2 Maximum and Minimum on an Interval (Extreme Values)

a) $y = x^3 - 5x^2 + 10, \; -5 \le x \le 5$ → Yes, this is a continuous function

b) $y = \frac{3x}{x-2}, \; -1 \le x \le 3$ → No, there is a discontinuity at $x = 2$.

c) $y = \frac{x}{x^2 - 4}, \; x \in [0, 5]$ → No, there is a discontinuity at $x = 2$.

d) $y = \frac{x^2 - 1}{x + 3}, \; x \in [-2, 3]$ → yes, it is continuous on the interval given.

2a) Max: $y = 8$ Min: $y = -12$

b) Max: $g(x) = 30$ Min: $g(x) = -5$

c) Max: $f(t) = 100$ Min: $f(t) = -100$

d) Max: $h(x) = 30$ Min: $h(x) = -20$

3a) $f(x) = x^2 - 4x$

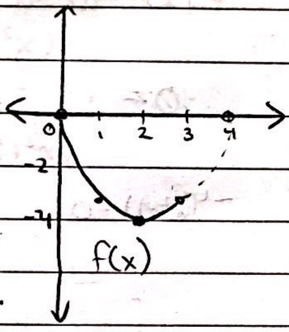
$f'(x) = 2x - 4$

$2x - 4 = 0$

$x = 2$ Max

$f(0) = 0$, $f(3) = -3$,

$f(2) = -4$ ← min.



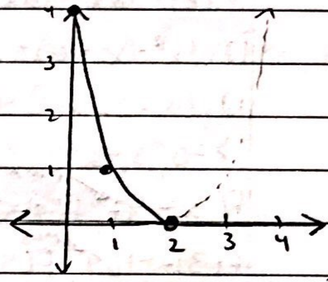
b) $f(x) = (x-2)^2$

$f'(x) = 2(x-2)$

$2x - 4 = 0$

$x = 2$

$\underset{\text{Max}}{f(0) = 4}$, $\underset{\text{Min.}}{f(2) = 0}$



c) $f(x) = x^3 - 3x^2$

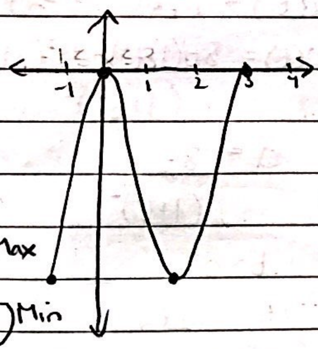
$f'(x) = 3x^2 - 6x$

$3x(x-2) = 0$

$x = 0, \; x = 2$

$f(-1) = -4$, $f(3) = 0$, Max

$f(0) = 0$, $f(2) = -4$ Min



d) $f(x) = x^3 - 3x^2, \; x \in [-2, 1]$

$f'(x) = 3x^2 - 6x$

$f(-2) = -20$ ← Min

$f(1) = -2$

$f(0) = 0$ ← Max

e) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

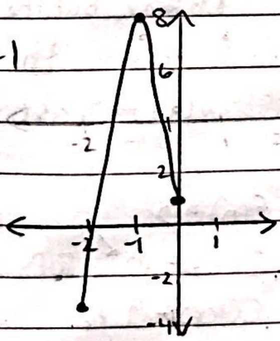
$6(x^2 - x - 2) = 0$

$6(x-2)(x+1) = 0$

$x = 2, \; x = -1$ Max

~~$f(2) = -19$~~, $f(-1) = 8$

$\underset{\text{Min}}{f(-2) = -3}$, $f(0) = 1$



f) $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$

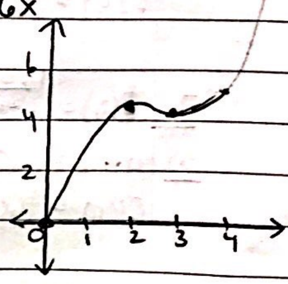
$f'(x) = x^2 - 5x + 6$

$(x-3)(x-2) = 0$

Min $x = 3, \; x = 2$

$\boxed{f(0) = 0}$ $f(2) = \frac{14}{3}$

$f(3) = \frac{9}{2}$, $\underset{\text{Max}}{\boxed{f(4) = \frac{16}{3}}}$



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4a) $f(x) = x + \frac{4}{x}$

$f'(x) = 1 - \frac{4}{x^2}$

$\frac{4}{x^2} = 1$

$4 = x^2$

$\pm 2 = x$

$\boxed{f(1) = 5}$ $f(2) = 4$ Min

$\boxed{f(10) = 10\frac{2}{5}}$ Max

b) $f(x) = 4\sqrt{x} - x,\ x \in [2, 9]$

$f'(x) = \frac{2}{\sqrt{x}} - 1$

$\frac{2}{\sqrt{x}} = 1$

$2 = \sqrt{x}$

$4 = x$

$f(2) = 4\sqrt{2} - 2$

$f(4) = 4$ ← Max

$f(9) = 3$ ← Min

c) $f(x) = \frac{1}{x^2 - 2x + 2},\ 0 \le x \le 2$

$f'(x) = \frac{-(2x-2)}{(x^2-2x+2)^2}$

$\frac{-2x+2}{(x^2-2x+2)^2} = 0$

$x = 1$

$f(0) = \frac{1}{2}$ ← Min

$f(1) = 1$ ← max

$f(2) = \frac{1}{2}$

d) $f(x) = 3x^4 - 4x^3 - 36x^2 + 20,\ x \in [-3, 4]$

$f'(x) = 12x^3 - 12x^2 - 72x$

$12x(x^2 - x - 6) = 0$

$12x(x-3)(x+2) = 0$

$x = 0, 3, -2$

$f(-3) = 47$ Max, $f(-2) = -44$, $f(0) = 20$

$f(3) = -169$ Min, $f(4) = -44$

e) $f(x) = \frac{4x}{x^2+1},\ -2 \le x \le 4$

$f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2}$

$0 = \frac{-4x^2 + 4}{(x^2+1)^2}$

$-4(x^2 - 1) = 0$

$x = \pm 1$

$f(-2) = -\frac{8}{5}$, $f(4) = \frac{16}{17}$,

$f(-1) = -2$, $f(1) = 2$

min max

f) max: $(4, \frac{16}{17})$, min: $(2, \frac{8}{5})$

5a) $v(t) = \frac{4t^2}{4+t^3},\ 1 \le t \le 4$

$v'(t) = \frac{8t(t^3+4) - 4t^2(3t^2)}{(4+t^3)^2}$

$0 = \frac{-4t^4 + 32t}{(4+t^3)^2}$

$-4t(t^3 - 8) = 0$

$-4t(t-2)(t^2+2t+4) = 0$

$f(1) = \frac{4}{5}$, $f(2) = \frac{4}{3}$, $\cancel{f(0)=0}$

$f(4) = \frac{16}{17}$ min max

b) $v(t) = \frac{4t^2}{1+t^2},\ t \ge 0$

$v'(t) = \frac{8t(1+t^2) - 4t^2(2t)}{(1+t^2)^2}$

$0 = \frac{8t}{(1+t^2)^2}$

$t = 0$

$v(0) = 0$ (min)

No max, but the top 4 faster than the bottom, so $V(t) \to 4$, as $t \to \infty$.

6. $N(t) = 30t^2 - 240t + 500$

$0 \le t \le 7$

$N'(t) = 60t - 240$

$60(t-4) = 0$

$t = 4$

$N(0) = 500$ $N(4) = 20$

$N(7) = 290$

$\therefore$ 20 was the lowest number of bacteria/cm^3.

7. $E(v) = \frac{1600v}{v^2+6400}$

a) $0 \le v \le 100$

$E'(v) = \frac{1600(v^2+6400) - 1600v(2v)}{(v^2+6400)^2}$

$0 = \frac{-1600v^2 + 10\,240\,000}{(v^2+6400)^2}$

$-1600(v^2 - 6400) = 0$

$v = \pm 80$

$E(0) = 0$

$E(80) = 10$ max at 80km/h

$E(100) = 9.76$

b) $E(50) = 8.99$ max at 50km/h

c) $0 \le v < 80$

d) $80 < v \le 100$

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8. $C(t) = \dfrac{0.1t}{(t+3)}$

$C'(t) = \dfrac{0.1(t+3)^2 - 0.1t(2(t+3))}{(t+3)^4}$

$= \dfrac{0.1(t+3)[t+3-2t]}{(t+3)^4}$

$= \dfrac{-0.1t+0.3}{(t+3)^3} \rightarrow t=3$

$C(1) = \dfrac{0.1}{64} = \dfrac{1}{640}$ min $C(3) = \dfrac{0.3}{36}$

$C(6) = \dfrac{0.6}{81} = \dfrac{1}{135}$ max $\rightarrow = \dfrac{1}{120}$

9. $P(t) = 2t + \dfrac{1}{(162t+1)}, \; 0 \le t \le 1$

$P'(t) = 2 + \dfrac{(162t+1)(0) - 1(162)}{(162t+1)^2}$

$0 = \dfrac{2(162t+1)^2 - 162}{(162t+1)^2}$ $P(0) = 1$

$P(1) = 2\frac{1}{163}$

$162 = 2(162t+1)^2$ $P(\frac{4}{81}) = \frac{8}{81} + \frac{1}{9} = \frac{17}{81}$ min.

$\pm 9 = 162t + 1$

$t = \frac{4}{81}, \; \cancel{t = \frac{-5}{81}}$

10. $r(x) = \frac{1}{4}\left(\frac{4900}{x} + x\right)$

$r'(x) = \frac{1}{4}\left(-\frac{4900}{x^2} + 1\right)$

$0 = -\frac{1225}{x^2} + \frac{1}{4}$

$(1225)(4) = x^2$

$\pm 70 = x$

$r(30) = 48\frac{1}{3}$ $r(120) = 40\ 21$

$r(70) = 35$ L/100km

200km → 70L gas

$70 \times 1.15 = \$80.50$

11. $f(x) = 0.001x^3 - 0.12x^2 + 3.6x + 10, \; 0 \le x \le 75$

$f'(x) = 0.003x^2 - 0.24x + 3.6$

$0.003(x^2 - 80x + 1200) = 0$

$0.003(x-60)(x-20) = 0$

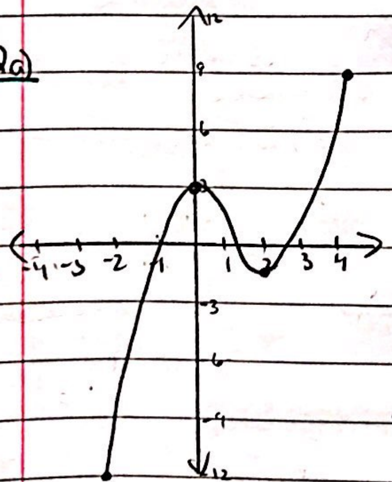
$x = 60, 20$

$f(0) = 10$ $f(20) = 42$ max $f(60) = 10$ min.

$f(75) = 26.875$

13. Turning point + endpoint

12a)



b) $-2 \le x \le 4$

c) $\uparrow x \in [-2, 0), (2, 4]$

$\downarrow x \in (0, 2)$

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