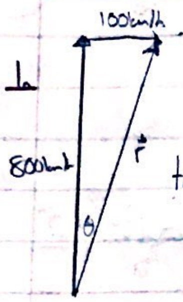
Chapter 7: Applications of Vectors

Review of Prerequisite Skills

1.



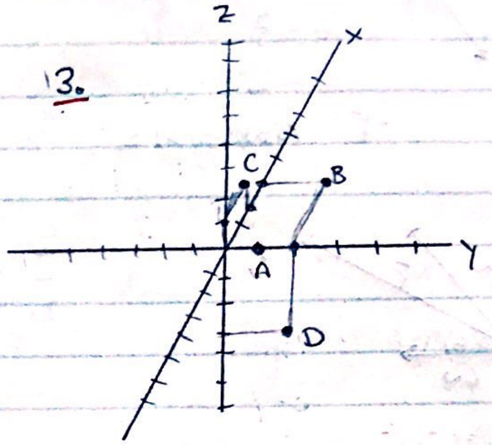
$|\vec{r}| = \sqrt{800^2 + 100^2}$

$= 806.23 \text{ km/h [N } 7.1° \text{ E]}$

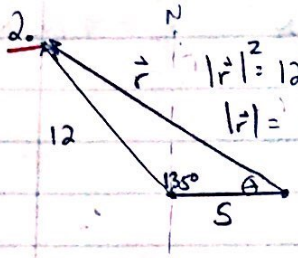
$\tan\theta = \frac{1}{8}$

$\theta = 7.1°$

13.



2.



$|\vec{r}|^2 = 12^2 + 5^2 - 2(12)(5)\cos 135°$

$|\vec{r}| = 15.93$ units [W 32.2° N]

$\frac{\sin\theta}{12} = \frac{\sin 135°}{15.93}$

$\theta = 32.2°$

5a) On the xy-plane

b) On the xz-plane

c) On the yz-plane

4a) $|(3,-2,7)|$
$= \sqrt{9+4+49}$
$= \sqrt{62}$

b) $|(-9,3,14)|$
$= \sqrt{81+9+196}$
$= \sqrt{286}$

c) $|(1,1,0)|$
$= \sqrt{2}$

d) $|(2,0,-9)|$
$= \sqrt{4+81}$
$= \sqrt{85}$

6a) $(-6,0) + 7(1,-1)$
$= (-6+7, 0-7)$
$= (1,-7)$

b) $(4,-1,3) - (-2,1,3)$
$= (6,-2,0)$

c) $2(-1,1,3) + 3(-2,3,-1)$
$= (-2,2,6) + (-6,9,-3)$
$= (-8,11,3)$

d) $-\frac{1}{2}(4,-6,8) + \frac{3}{2}(4,-6,8)$
$= (4,-6,8)$

7a) $3\vec{i} - 2\vec{j} - \vec{k} + (-2\vec{i} + \vec{j})$
$= \vec{i} + 3\vec{j} - \vec{k}$

b) $3\vec{i} + 2\vec{j} - \vec{k} - (-2\vec{i} + \vec{j})$
$= 5\vec{i} + \vec{j} - \vec{k}$

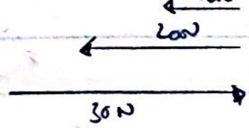
c) $2(3\vec{i} + 2\vec{j} - \vec{k}) - 3(-2\vec{i} + \vec{j})$
$= 6\vec{i} + 4\vec{j} - 2\vec{k} + 6\vec{i} - 3\vec{j}$
$= 12\vec{i} + \vec{j} - 2\vec{k}$

7.1 Vectors as Forces

1a) 10 N - melon, squash
50 N - chair, small table
100 N - coffee table

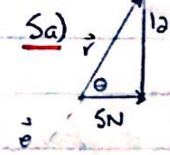
b) $63 \text{ kg} \times 9.8 \text{ m/s}^2 = 617.4 \text{ N}$

2a)



b) 180°

5a)



$|\vec{r}| = \sqrt{169}$

$\vec{F} = 13 \text{ N [N } 22.6° \text{ E]}$

$\tan\theta = \frac{12}{5}$

$\theta = 67.4°$

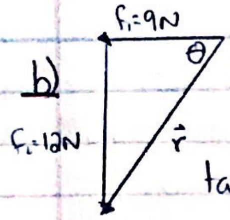
$-\vec{F} = 13 \text{ N [S } 22.6° \text{ W]}$

3. On the same side in the same direction.

4. A triangle is 2-D, so the forces must be in the same plane.

Scanned with CamScanner

b)



$|\vec{r}| = \sqrt{225}$

$\vec{F} = 15N$ [S 37° W]

$\tan\theta = \frac{4}{3}$

$\theta = 53°$

$-\vec{F} = 15N$ [N 37° E]

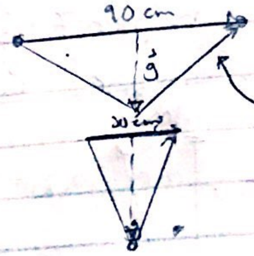
6a) yes

b) yes

c) no - no Δ!

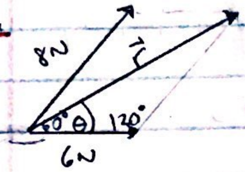
d) yes

7.



resultant is smaller, meaning less force to aid in the pull-up.

8.



$|\vec{r}| = 6^2 + 8^2 - 2(6)(8)\cos 120°$

$= 12.17N$

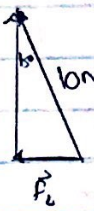
$\frac{\sin\theta}{8} = \frac{\sin 120°}{12.17}$

$\theta = 34.7°$

$\vec{r} = 12.17N$ 34.7° from $\vec{f_1}$ toward $\vec{f_2}$

$-\vec{r} = 12.17N$ 145.3° from $\vec{f_1}$

9.



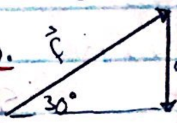
$\sin 15° = \frac{\vec{f_2}}{10}$

$\vec{f_2} = 2.59N$

$\cos 15° = \frac{\vec{f_1}}{10}$

$\vec{f_1} = 9.66N$

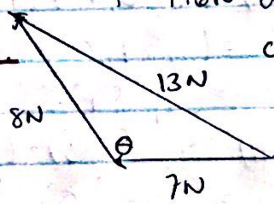
10.



$\sin 30° = \frac{98}{\vec{f}}$

$\vec{f} = 196N$ up the ramp

11.

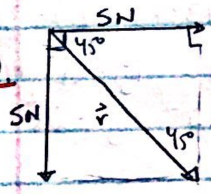


$\cos\theta = \frac{13^2 - 7^2 - 8^2}{-2(7)(8)}$

$\theta = -\frac{1}{2}$

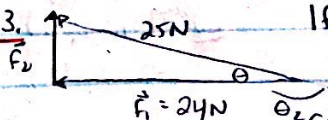
$\theta = 120°$

12.



$\vec{r} = 5\sqrt{2}$ [S 45° E]

13.



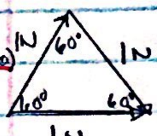
$|\vec{f_2}| = \sqrt{25^2 - 24^2}$

$= 7N$

$\cos\theta = \frac{24}{25}$

$\theta = 16.3°$ $\theta_2 = 163.7°$

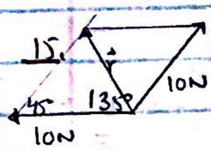
14a)



1N equilateral Δ, so all angles 60°

b) $180 - 60 = 120°$

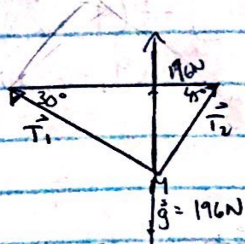
15.



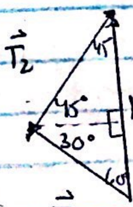
$|\vec{r}|^2 = 10^2 + 10^2 - 2(10)(10)\cos 45°$

$\vec{r} = 7.65$ N 67.5° W

16.



$\vec{T_1} + \vec{T_2} = 196N$



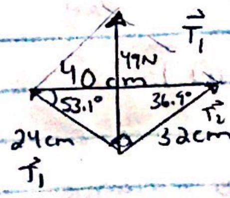
$\frac{\vec{T_1}}{\sin 45°} = \frac{196}{\sin 75°}$

$\vec{T_1} = 143.5N$

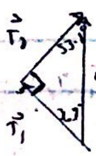
$\frac{\vec{T_2}}{\sin 60°} = \frac{196}{\sin 75°}$

$\vec{T_2} = 175.7N$

17.



$\vec{T_1} + \vec{T_2} = 49N$



$\vec{T_1} = 49\sin 53.1°$

$\vec{T_1} = 39.18N$

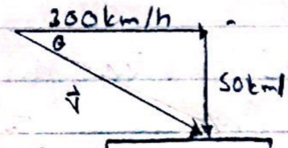
$\vec{T_2} = 49\sin 36.9°$

$= 29.4N$

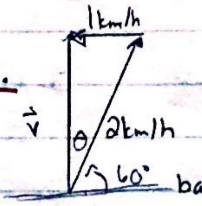
7.2 Velocity

1a) 84 km/h in the direction of the train
b) 76 km/h in the direction of the train

2a) 500 km/h N
b) 700 km/h N

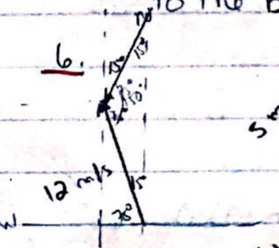
3.


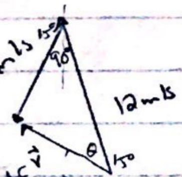
$|\vec{v}|^2 = \sqrt{300^2 + 50^2}$
$|\vec{v}| = 304.14$ km/h
$\tan\theta = \frac{1}{6}$
$\theta = 9.5°$
$\vec{v} = 304.14$ km/h W 9.5° S

4.


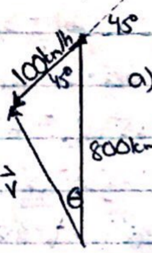
$\sin\theta = \frac{1}{2}$
$\theta = 30°$
∴ he has to swim at a 60° angle to the bank.

5a) 2 m/s
b) 22 m/s

6.


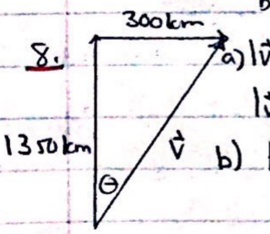


$|\vec{v}|^2 = 12^2 + 5^2$
$|\vec{v}| = 13$ m/s
$\sin\theta = \frac{5}{12}$
$\theta = 24.6°$
$\vec{v} = 13$ m/s N 39.6° W

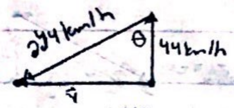
7.


a) $|\vec{v}|^2 = 100^2 + 800^2 - 2(100)(800)\cos 45°$
$|\vec{v}| = 732.7$ km/h
$\frac{\sin\theta}{100} = \frac{\sin 45°}{732.7}$
$\theta = 5.5°$
$\vec{v} = 732.7$ km/h N 5.5° W

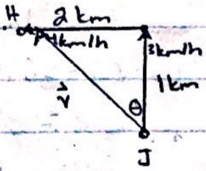
b) 732.7 km

8.


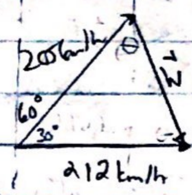
a) $|\vec{v}| = \sqrt{1350^2 + 300^2}$
$|\vec{v}| = 1382.9$ km
b) $\tan\theta = \frac{30}{135}$
$\theta = 12.5°$
N 12.5° E

9


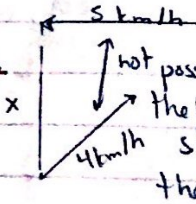
a) $\cos\theta = \frac{44}{244}$
$\theta = 79.6°$
W 10.4° S
b) $|\vec{v}|^2 = 244^2 + 44^2$
$|\vec{v}| = 247.9$ km/h
$480 \div 247.9 = 1.94$ hours

10.


a) $|\vec{v}| = \sqrt{3^2 + 4^2}$
$= 5$ km/h
$\tan\theta = 4/3$
$\theta = 53.1°$
$\vec{v} = 5$ km/h 53.1° downstream
b) $4 \div 3 = 1.33$ km travelled
∴ She is 0.67 km from Helen's
c) 20 minutes

11.


$|\vec{w}|^2 = 205^2 + 212^2 - 2(205)(212)\cos 30°$
b) $|\vec{w}| = 108.1$ km/h
a) $\frac{\sin\theta}{212} = \frac{\sin 30°}{108.1}$
$\theta = 78.7°$
$\boxed{\text{S } 18.7° \text{ E}}$

12.


not possible. the hyp. is shorter than the side.

Hilary

7.3 The Dot Product of Two Geometric Vectors

1. Both $\vec{a}$ and $\vec{b}$ must be non-zero vectors.
2. $\vec{a}\cdot\vec{b}$ yields a scalar quantity. You can't find the dot product of a scalar and a vector.
3. $|\vec{a}|=(1,0)$, $\vec{b}=(2,0)$, $\vec{c}=(-1,0)$

$\vec{a}\cdot\vec{b}=(1)(2)\cos 90^\circ$
$=0$

$\vec{b}\cdot\vec{c}=(1)(2)\cos 90^\circ$
$=0$

$\therefore\ \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}$, but $\vec{a}\neq\vec{c}$.

4. If $\vec{a}=\vec{c}$, and $\vec{b}=\vec{b}$, it follows that $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}$.

5. $\vec{a}\cdot\vec{b}=(1)(1)\cos 180^\circ$
$\vec{a}\cdot\vec{b}=-1$

6a) $\vec{p}\cdot\vec{q}=(4)(8)\cos 60^\circ$
$\vec{p}\cdot\vec{q}=16$

b) $\vec{x}\cdot\vec{y}=(2)(4)\cos 150^\circ$
$=-4\sqrt{3}$

c) $\vec{a}\cdot\vec{b}=(0)(8)\cos 100^\circ$
$=0$

d) $\vec{p}\cdot\vec{q}=(1)(1)\cos 180^\circ$
$=-1$

e) $\vec{m}\cdot\vec{n}=(2)(5)\cos 90^\circ$
$=0$

f) $\vec{u}\cdot\vec{v}=(4)(8)\cos 145^\circ$
$=-26.2$

7a) $\cos\theta=\dfrac{12\sqrt{3}}{(8)(3)}$

$\cos\theta=\dfrac{\sqrt{3}}{2}$

$\theta=30^\circ$

b) $\cos\theta=\dfrac{6}{(6)(6)}$

$\theta=80.4^\circ$

c) $\cos\theta=\dfrac{3}{(1)(5)}$

$\theta=\dfrac{3}{5}$

$\theta=53.1^\circ$

7d) $\cos\theta=\dfrac{-3}{(1)(5)}$

$\theta=126.9^\circ$

e) $\cos\theta=\dfrac{10.5}{(7)(3)}$

$\theta=60^\circ$

f) $\cos\theta=\dfrac{-50}{(10)(10)}$

$\cos\theta=-\frac{1}{2}$

$\theta=120^\circ$

8. $\vec{a}\cdot\vec{b}=(7.5)(6)\cos 60^\circ$
$=22.5$

9a) $(\vec{a}+5\vec{b})\cdot(2\vec{a}-3\vec{b})$
$=\vec{a}\cdot 2\vec{a}+\vec{a}\cdot(-3\vec{b})+5\vec{b}\cdot 2\vec{a}+5\vec{b}\cdot(-3\vec{b})$
$=2|\vec{a}|^2-3(\vec{a}\cdot\vec{b})+10(\vec{a}\cdot\vec{b}),-15|\vec{b}|^2$
$=2|\vec{a}|^2-15|\vec{b}|^2+7(\vec{a}\cdot\vec{b})$

b) $3\vec{x}\cdot(\vec{x}-3\vec{y})-(\vec{x}-3\vec{y})\cdot(-3\vec{x}+\vec{y})$
$=3(\vec{x}\cdot\vec{x})-9(x\cdot y)-(-3(\vec{x}\cdot x)+\vec{x}\cdot\vec{y}+9(x\cdot y)-3(y\cdot y))$
$=3|\vec{x}|^2+3|\vec{x}|^2+3|\vec{y}|^2-10(\vec{x}\cdot\vec{y})-9(\vec{x}\cdot\vec{y})$
$=6|\vec{x}|^2+3|\vec{y}|^2-19(x\cdot y)$

10. Zero, because $|\vec{0}|=0$.

11. $(\vec{a}-5\vec{b})\cdot(\vec{a}-\vec{b})=0$, $|\vec{a}|=1$, $|\vec{b}|=1$
$\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-5\vec{a}\cdot\vec{b}+5(b\cdot b)=0$
$|\vec{a}|^2-6(\vec{a}\cdot\vec{b})+5|\vec{b}|^2=0$
$|\vec{a}|^2+5|\vec{b}|^2=6(\vec{a}\cdot\vec{b})$
$1+5=6(\vec{a}\cdot\vec{b})$
$1=\vec{a}\cdot\vec{b}$

12a) LS $=(\vec{a}+\vec{b})(\vec{a}+\vec{b})$ $\quad\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

$|\vec{a}|^2\longrightarrow\ =\vec{a}\cdot\vec{a}+\boxed{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}}+\vec{b}\cdot\vec{b}$
$=|\vec{a}|^2+2(\vec{a}\cdot\vec{b})+|\vec{b}|^2$
$\therefore$ LS = RS

b) LS $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$
$=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-b\cdot b$
$=|\vec{a}|^2-|\vec{b}|^2$
$\therefore$ LS = RS

13a) $|\vec{a}|^2=\vec{b}\cdot\vec{b}+2(\vec{b}\cdot\vec{c})+\vec{c}\cdot\vec{c}$
$|\vec{a}|^2=(\vec{b}+\vec{c})\cdot(\vec{b}+\vec{c})$
and $\vec{a}=\vec{b}+\vec{c}$

b) $|\vec{a}|^2=|\vec{b}|^2+2(\vec{b}\cdot\vec{c})+|\vec{c}|^2$
$\uparrow$
$=0$ if $b\perp c$

$\therefore\ |\vec{a}|^2=|\vec{b}|^2+|\vec{c}|^2$
Pyth. theorem.

14. $(\vec u+\vec v+\vec w)\cdot(\vec u+\vec v+\vec w)$

$=|\vec u|^2+\cancel{\vec u\cdot\vec v}+\cancel{\vec u\cdot\vec w}+\vec u\cdot\vec v+|\vec v|^2+\vec v\cdot\vec w+\vec u\cdot\vec w+\vec v\cdot\vec w+|\vec w|^2$

$=(1)^2+(2)^2+(3)^2$

$=14$

15. $LS=|\vec u+\vec v|^2+|\vec u-\vec v|^2$ $\qquad RS=2|\vec u|^2+2|\vec v|^2$

$=(\vec u+\vec v)\cdot(\vec u+\vec v)+(\vec u-\vec v)\cdot(\vec u-\vec v)$

$=|\vec u|^2+\vec u\cdot\vec v+\cancel{\vec u\cdot\vec v}+|\vec v|^2+|u|^2-\cancel{\vec u\cdot\vec v}-\vec u\cdot\vec v+|\vec v|^2$

$=2|\vec u|^2+2|\vec v|^2$ $\quad\therefore LS=RS.$

16. $(\vec a+\vec b)\cdot(\vec a+\vec b+\vec c)$

$=|\vec a|^2+\vec a\cdot\vec b+\vec a\cdot\vec c+\vec a\cdot\vec b+|\vec b|^2+\vec b\cdot\vec c$

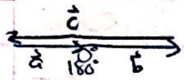
$=|\vec a|^2+|\vec b|^2+2a\cdot b+a\cdot c+b\cdot c$

$=1+1+2(1)(1)\cos60^\circ+(1)(1)\cos60^\circ+(1)(1)\cos60^\circ$

$=2+1+\frac{1}{2}+\frac{1}{2}$

$=3$

17.



$\vec a\cdot\vec b+\vec a\cdot\vec c+\vec b\cdot\vec c$

$=|\vec a||\vec b|\cos\theta+|\vec a||\vec c|\cos\beta+|\vec b||\vec c|\cos\theta$

$=(1)(2)(-1)+(1)(3)(-1)+(2)(3)(-1)$

$=2-3-6$

$=-7$

7.4 The Dot Product of Algebraic Vectors

1. Any vector with ⊕ components (2, 2), (3, 3), etc.

2a) $\vec a\cdot\vec b=(-2)(1)+(1)(2)$

$=0$, $90°$

b) $\vec a\cdot\vec b=(2)(4)+(3)(3)+(-1)(-17)$

$=8+9+17$

$=34$, acute

c) $\vec a\cdot\vec b=(1)(3)+(-2)(-2)+(5)(-2)$

$=3+4-10$

$=-3$, obtuse

3a) $(0,0,1)$

b) $(0,1,0)$

c) $(1,0,0)$

4a) $(1,2,-1)\perp(4,3,10)$, $(-4,-5,-6)\perp(5,-3,-\frac{5}{6})$

b) No, none are scalar multiples.

5a) The only way to have a vector ⊥ to the xy-plane is to move in the z-direction.

b) Same as (a)

6a) $\cos\theta=\dfrac{(5)(-1)+(3)(-2)}{(\sqrt{34})(\sqrt5)}$

$\theta=148°$

6b) $\cos\theta=\dfrac{(-1)(6)+(4)(-2)}{(\sqrt{17})(\sqrt{40})}$

$=-\dfrac{14}{\sqrt{680}}$

$\theta=122°$

c) $\cos\theta=\dfrac{(2)(2)+(2)(1)+(1)(-2)}{(3)(3)}$

$=\dfrac{4}{9}$

$\theta=64°$

d) $\cos\theta=\dfrac{(2)(-5)+(3)(0)+(-6)(12)}{(7)(13)}$

$=-\dfrac{82}{91}$

$\theta=154°$

7a) $\vec{a}\cdot\vec{b} = 6k - 2 - 3k$

$3k - 2 = 0 \leftarrow \theta = 90°$

$k = 2/3$

b) $\vec{a}\cdot\vec{b} = 0 + k$

$= k$

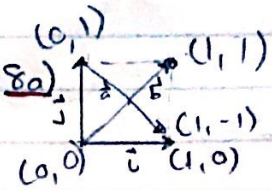
$\frac{\sqrt{2}}{2} = \frac{k}{\sqrt{2}k}$

$2k = 2k$

$k = k$

$k \geq 0$

8a)



b) $(1,1)$ and $(1,-1)$

c) $\vec{a}\cdot\vec{b}$

$= (1,1)\cdot(1,-1)$

$= 1 - 1$

$= 0$ $\therefore$ The angle is $90°$.

9a) $\vec{a}\cdot\vec{b} = (1-\sqrt{2}, \sqrt{2}, -1)\cdot(1,1)$

$= (1-\sqrt{2}) + \sqrt{2} - 1$

$= 0$

$\cos\theta = 0$, so $\theta = 90°$

b) $\vec{a}\cdot\vec{b} = (\sqrt{2}-1, \sqrt{2}+1, \sqrt{2})\cdot(1,1,1)$

$= \sqrt{2} - 1 + \sqrt{2} + 1 + \sqrt{2}$

$= 3\sqrt{2}$

$|\vec{a}| = \sqrt{(\sqrt{2}-1)^2 + (\sqrt{2}+1)^2 + (\sqrt{2})^2}$

$= \sqrt{2 - \cancel{2\sqrt{2}} + 1 + 2 + \cancel{2\sqrt{2}} + 1 + 2}$

$= 2\sqrt{2}$

$|\vec{b}| = \sqrt{3}$

$\cos\theta = \frac{3\sqrt{2}}{(2\sqrt{2})(\sqrt{3})}$

$= \frac{3}{2\sqrt{3}}$ or $\frac{\sqrt{3}}{2}$

$\theta = 30°$

10.i) $8 \times \frac{3}{2} = 12$

$\therefore p = 4 \times \frac{2}{3}$

$= \frac{8}{3}$

$q = 2 \times \frac{3}{2}$

$= 3$

ii) $(2, p, 8)\cdot(q, 4, 12) = 0$

$2q + 4p + 96 = 0$

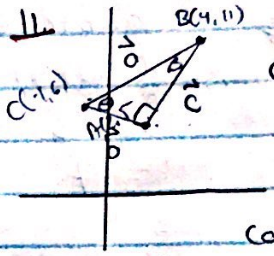
$q + 2p = -48$

$q = -48 - 2p$

All values that satisfy this equation!

b) Yes, when looking for collinear vecters.
No when looking at $\perp$ (set of answers)

11.



$\cos\theta = \frac{15 - 5}{(\sqrt{58})(\sqrt{50})}$

$= 63.4°$

$\cos B = \frac{10 + 30}{(\sqrt{40})(\sqrt{50})}$

$= 26.6°$

$\angle A = 90°$.

$\vec{a} = (5, 5)$

$\vec{b} = (3, -1)$

$\vec{c} = (-2, -6)$

12a) $B(7,4,0)$

$A(7,0,0)$

$C(0,4,0)$

$D(7,0,5)$

$E(0,4,5)$

$F(0,0,5)$

b) $\vec{AE} = (-7, 4, 5)$

$\vec{BF} = (-7, -4, 5)$

$|\vec{AE}| = 3\sqrt{10} = |\vec{BF}|$

$\cos\theta = \frac{49 - 16 + 25}{(3\sqrt{10})^2}$

$= \frac{58}{90}$

$\theta = 50°$

14. $\vec{r}\cdot\vec{s} = 0$

$(p, p, 1)\cdot(p, -2, -3) = 0$

$p^2 - 2p - 3 = 0$

$(p-3)(p+1) = 0$

$p = 3$ or $p = -1$

13a) $(-1, 3, 0)\cdot(a, b, c) = 0$

$-a + 3b = 0 \rightarrow a = 3b$

$(1, -5, 2)\cdot(a, b, c) = 0$

$a - 5b + 2c = 0$

$3b - 5b + 2c = 0$

$-2b + 2c = 0$

$c = b$

Let $a = 3$, $b = 1$, $(3, 1, 1)$.

b) $(1, 3, -4)\cdot(a, b, c) = 0$

$a + 3b - 4c = 0$

$(-1, -2, 3)\cdot(a, b, c) = 0$

$-a - 2b + 3c = 0 \rightarrow a = -2b + 3c$

$-2b + 3c + 3b - 4c = 0$

$b - c = 0$

$b = c$

If $b = c = 1$

$a = 1$

$(1, 1, 1)$

$-2\vec{r} = \vec{s}$

15a) $\vec{c} \cdot \vec{d} = 0$

$(-3, p, -1) \cdot (1, -4, q) = 0$

$-3 - 4p - q = 0$

$q = -3 - 4p$

b) $-3 = -3 - 4p$

$0 = -4p$

$0 = p$

16. $(1, 2, -1) \cdot (a, b, c) = 0$ $(-2, -4, 2) \cdot (a, b, c)$

$a + 2b - c = 0$ ← ⊥ to both!

Let $b = 1$, $c = 1$, so $a = -1$

$(-1, 1, 1)$

Let $a = 1$, $c = 1$, so $b = 0$

$(1, 0, 1)$ etc.

17. $\vec{x} \cdot \vec{y} = 8 + 3p - 12$

$= 3p - 4$

$|\vec{x}| = \sqrt{20 + p^2}$

$|\vec{y}| = 7$

$\cos\theta = \frac{4}{21}$

$\therefore \frac{4}{21} = \frac{3p - 4}{7\sqrt{20 + p^2}}$

$28\sqrt{20 + p^2} = 63p - 84$

$\sqrt{20 + p^2} = \frac{9}{4}p - 3$

$20 + p^2 = \frac{81}{16}p^2 - \frac{27}{2}p + 9$

$\frac{65}{16}p^2 - \frac{27}{2}p - 11 = 0$

$65p^2 - 216p - 176 = 0$

$p = \frac{216 \pm \sqrt{216^2 - 4(65)(-176)}}{2(65)}$

$p = \frac{216 \pm 304}{130}$

$= 4$, $p = -\frac{44}{65}$

Mid-Chapter Review

1a) $\vec{a} \cdot \vec{b} = (3)(2)\cos 60°$

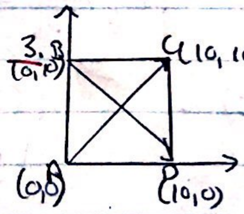
$= 3$

b) $(3\vec{a} + 2\vec{b}) \cdot (4\vec{a} - 3\vec{b})$

$= 12|a|^2 - 9(\vec{a} \cdot \vec{b}) + 8(\vec{a} \cdot \vec{b}) - 6|b|^2$

$= 12(3)^2 - 3 - 6(2)^2$

$= 81$



$\vec{AC} = (10, 10)$

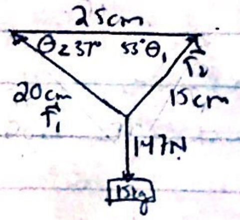
$\vec{BD} = (10, -10)$

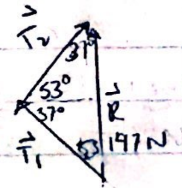
$\vec{AC} \cdot \vec{BD} = (10, 10) \cdot (10, -10)$

$= 100 - 100$

$= 0$

2.





$\cos\theta_2 = \frac{15^2 - 25^2 - 20^2}{-2(25)(20)}$

$\theta_2 = 37°$

$\cos\theta_1 = \frac{20^2 - 25^2 - 15^2}{-2(25)(15)}$

$\theta_1 = 53°$

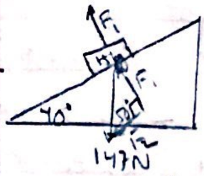
$\frac{\vec{T_1}}{\sin 37°} = \frac{147}{\sin 90°}$

$\vec{T_1} = 88.5\,N$

$\frac{\vec{T_2}}{\sin 53°} = \frac{147}{\sin 90°}$

$\vec{T_2} = 117.4\,N$

5.



a) $\sin 50° = \frac{\vec{F_1}}{147}$

$\vec{F_1} = 112.61N$

b) $\cos 50° = \frac{\vec{F_2}}{147}$

$\vec{F_2} = 94.49N$

6. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$|\vec{a}| = |\vec{b}| = 3$

Sum of angles: 900

$\theta = 150°$

$\therefore \vec{a} \cdot \vec{b} = 9\cos 150°$

$= -\frac{9\sqrt{3}}{2}$

7a) $\vec{a} \cdot \vec{b} = 4 - 10 + 40$

$= 34$

b) $|\vec{a}| = 21 \quad |\vec{b}| = 3$

$\vec{c} = (-3, 7, -18)$

$|\vec{c}| = \sqrt{382}$

$\cos\theta = \frac{382 - 441 - 9}{-2(21)(3)}$

$= \frac{34}{63}$

8a) $\vec{a} \cdot \vec{b}$

$= (1, 2, 1) \cdot (2, -3, 4)$

$= 2 - 6 + 4$

$= 0$

8b) $\vec{b} \cdot \vec{c}$

$= (2, -3, 4) \cdot (3, -1, -1)$

$= 6 + 3 - 4$

$= 5$

c) $\vec{b} + \vec{c}$

$= 2\vec{i} - 3\vec{j} + 4\vec{k} + 3\vec{i} - \vec{j} - \vec{k}$

$= 5\vec{i} - 4\vec{j} + 3\vec{k}$

d) $\vec{a} \cdot (\vec{b} + \vec{c})$

$= (1, 2, 1) \cdot (5, -4, 3)$

$= 5 - 8 + 3$

$= 0$

e) $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c})$

$= (3, -1, 5) \cdot (5, -4, 3)$

$= 15 + 4 + 15$

$= 34$

f) $(2\vec{a} - 3\vec{b}) \cdot (2\vec{a} + \vec{c})$

$= (2\vec{i} + 4\vec{j} + 2\vec{k} - (6\vec{i} - 9\vec{j} + 12\vec{k})) \cdot (2\vec{i} + 4\vec{j} + 2\vec{k} + 3\vec{i} - \vec{j} - \vec{k})$

$= (-4, 13, -10) \cdot (5, 3, 1)$

$= -20 + 39 - 10$

$= 9$

9a) $\vec{p} \cdot \vec{q} = 0$

$3x^2 + 10x + 3 = 0$

$(3x + 1)(x + 3) = 0$

$\therefore x = -\frac{1}{3}, -3$

b) $p = k\vec{q}$

- not possible because $k = \frac{1}{3}$ for $x\vec{i}$ to $= k(3x\vec{i})$ and $k = \frac{1}{10}x$ for $\vec{j} = 10x\vec{j}$

10a) $3\vec{x} - 2\vec{y}$

$= 3\vec{i} - 6\vec{j} - 3\vec{k} - (2\vec{i} - 2\vec{j} - 2\vec{k})$

$= \vec{i} - 4\vec{j} - \vec{k}$

b) $3\vec{x} \cdot 2\vec{y}$

$= (3, -6, -3) \cdot (2, -2, -2)$

$= 6 + 12 + 6$

$= 24$

c) $|\vec{x} - 2\vec{y}|$

$= |(1, -2, -1) - (2, -2, -2)|$

$= |(-1, 0, 1)|$

$= \sqrt{2}$

10d) $(2\vec{x} - 3\vec{y}) \cdot (x + 4\vec{y})$

$= (-1, -1, 1) \cdot (5, -6, -5)$

$= -5 + 6 - 5$

$= -4$

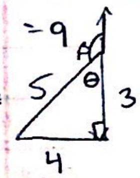
e) $(2, -4, -2) \cdot (1, -2, -1) - (5, -5, -5) \cdot (1, -2, -1)$

$= 2 + 4 + 2 - (5 + 10 + 5)$

$= 8 - 20$

$= -12$

11.



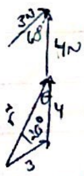
$\cos\theta = \frac{3}{5}$

(Pyth. triple) $\theta = 53.1°$

$\beta = 180 - 53.1°$

$= 126.9°$

12



a) $\frac{\sin\theta}{3} = \frac{\sin 120°}{\sqrt{37}}$

$\theta = 25.3°$

b) $\vec{E} = 6.08N$ 154.7° from the 4N force

$|\vec{r}| = 4^2 + 3^2 - 2(4)(3)\cos 120°$

$|\vec{r}| = \sqrt{37}$ or 6.08N, 25.3° from 4N force.

13 $\vec{p} = (3, -10, 0)$ $\quad |\vec{m}| = \sqrt{38}$ $\quad |\vec{n}| = \sqrt{75}$

$|\vec{p}| = \sqrt{109}$

b) $\cos\theta = \frac{109 - 38 - 75}{-2(\sqrt{38})(\sqrt{75})}$

$= 87.9°$

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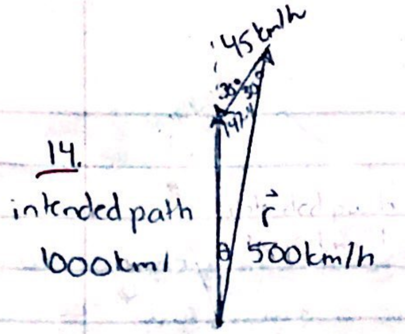
14.



a) $\frac{\sin\theta}{45} = \frac{\sin 30°}{500}$

$\theta = N2.6°E$

b) $\frac{|\vec{r}|}{\sin 147.4} = \frac{1000}{\sin 30°}$

$|\vec{r}| = 1077.5$ km

$1077.5 \div 500 = 2.16$ hours

15. $(x,y,z)\cdot(-1,2,5)=0$ and $(x,y,z)\cdot(1,3,5)=0$

$-x+2y+5z=0$ ① $x+3y+5z=0$ ②

①+② $\quad x+3y+5z=0$

$5y+10z=0$

sub $\boxed{y=-2z}$ into ①: $-x+2(-2z)+5z=0$

$x=z$

$\therefore \vec{x}=(t, -2t, t)$

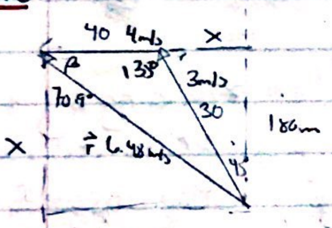
unit vector: $\frac{1}{|\vec{x}|}(t,-2t,t)$

BOB: Let $t=1$
$|\vec{x}|=\sqrt{6}$,
so $\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

17a) When $\vec{x}$ and $\vec{y}$ are equal in magnitude.

b) $\vec{a}+\vec{b}$ is one diagonal (long) and $(\vec{a}-\vec{b})$ is the other (short), and $\vec{a}$ and $\vec{b}$ are the sides.

16.



a) $\sin 45° = \frac{x}{30}$

$x = 21.2$m

Total $= 40+21.2$

$= 61.2$m downstream

b) $|\vec{r}|^2 = 3^2+4^2-2(3)(4)\cos 135°$

$|\vec{r}| = 6.48$ m/s

$\frac{\sin\beta}{3} = \frac{\sin 135°}{6.48}$

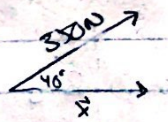
$\beta = 19.1°$

$\cos 70.9° = \frac{x}{6.48}$

2.12 m/s $= x$

$180 \div 2.12 = 84.9$ seconds.

18.



Let $\vec{x}$ be the horizontal component.

$\cos 40° = \frac{\vec{x}}{350}$

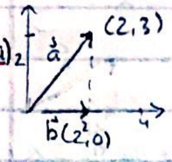
$x = 268.1$N

$\vec{a}$ on $\vec{b}$ $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

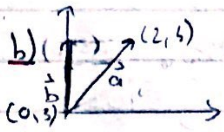
$\vec{b}$ on $\vec{a}$ $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

7.5 Scalar and Vector Projections

1a)



b)



$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{4}{2}$

$= 2$

vector projection

$\frac{4}{4} \cdot 2\vec{i}$

$= 2\vec{i}$

$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{9}{3}$

$= 3$

vector proj.

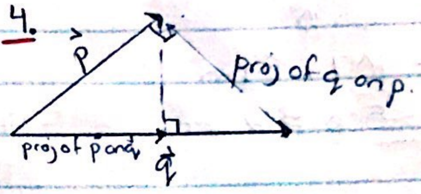
$\frac{9}{9} \cdot 3\vec{j}$

$= 3\vec{j}$

2. The vector that it is projecting on to must have a magnitude

3. When vectors are $\perp$, the scalar projection of one on to the other is 0 (and the vector is $\vec{0}$) because you are projecting directly on to the tail.

4.



Not the same!

5. $\vec{OP}$ on $\vec{i}$: $\frac{-1}{1}$

$= -1$

vector: $\frac{-1}{1}\vec{i}$

$= -1\vec{i}$

$\vec{OP}$ on $\vec{j}$: $\frac{2}{1}$

$= 2$

vector $\frac{2}{1}\vec{j}$

$= 2\vec{j}$

$\vec{OP}$ on $\vec{k}$: $\frac{-5}{1}$

$= -5$

vector: $\frac{-5}{1}\vec{k}$

$= -5\vec{k}$

*Dividing by 1 doesn't change the answer!

6a) $proj_{\vec{q}}\vec{p} = \frac{\vec{p}\cdot\vec{q}}{|\vec{q}|}$

$= \frac{-12+30+440}{21}$

$= 458/21$

vector $proj_{\vec{q}}\vec{p} = \frac{458}{441}(-4,5,20)$

b) $\cos\alpha = \frac{3}{23}$

$= 82.5°$

$\cos\beta = \frac{6}{23}$

$= 74.9°$

$\cos\gamma = \frac{-22}{23}$

$= 163.0°$

7a) $proj_{\vec{y}}\vec{x} = \frac{\vec{x}\cdot\vec{y}}{|\vec{y}|}$

$= 0$

vector $proj_{\vec{y}}\vec{x} = \vec{0}$

b) $proj_{\vec{y}}\vec{x} = \frac{2}{1}$

$= 2$

vector $proj_{\vec{y}}\vec{x} = 2(1,0)$

$= 2\vec{i}$

c) $proj_{\vec{y}}\vec{x} = \frac{50}{13}$

vector $proj_{\vec{y}}\vec{x} = \frac{50}{169}(5,12)$

8a) $proj_x\vec{a} = -1$ $proj_y\vec{a} = 2$ $proj_z\vec{a} = 4$

v. $proj_x\vec{a} = -1\vec{i}$ v. $proj_y\vec{a} = 2\vec{j}$ v. $proj_z\vec{a} = 4\vec{k}$

b) $-m, -m\vec{i}$ $2m, 2m\vec{j}$ $4m, 4m\vec{k}$

9a) $\xrightarrow{\vec{a}}$ projection of $\vec{a}$ on $\vec{a}$ is just $\vec{a}$

b) $\cos 0° = 1$, and $\frac{\vec{a}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} = \vec{a}$, so the results are true.

10a) $\xleftarrow{-\vec{a}} \xrightarrow{\vec{a}}$

$proj_{\vec{a}}-\vec{a} = -\vec{a}$

b) $\cos 180° = -1$, see above.

11a) $\vec{AB} = (-2, 1, 2)$

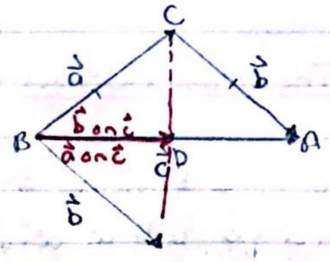
$proj_x \vec{AB} = -2$ $\quad proj_y \vec{AB} = 1$ $\quad proj_z \vec{AB} = 2$

v. proj $= -2\vec{i}$ $\quad$ v. proj. $= \vec{j}$ $\quad$ v. proj $= 2\vec{k}$

b) $\cos\beta = \dfrac{1}{3}$

$\beta = 70.5°$

12.



c) The scalar proj is the same because it is an isosceles Δ, so CD bisects AB.

d) Yes!

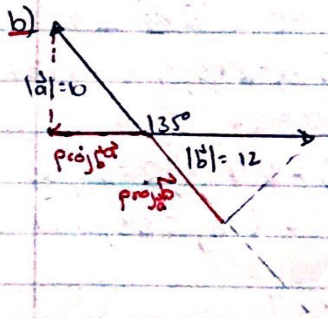
13. $\vec{a} \cdot \vec{b} = (10)(12)\cos 135°$
$= -60\sqrt{2}$

a) $proj_{\vec{b}} \vec{a} = \dfrac{-60\sqrt{2}}{12}$
$= -7.07$

$proj_{\vec{a}} \vec{b} = \dfrac{-60\sqrt{2}}{10}$
$= -8.49$

$\therefore proj_{\vec{b}} \vec{a} \neq proj_{\vec{a}} \vec{b}$

b)



14a) $\vec{AB}(-3, 2, -1)$

$proj_{\vec{OD}} \vec{AB} = \dfrac{-3+4-2}{3}$
$= \dfrac{-1}{3}$

b) $\vec{BC} = (-7, 4, 2)$ $\quad \vec{AC} = (-4, 6, 1)$

$proj_{\vec{OD}} \vec{BC} = \dfrac{7+8+4}{3}$ $\quad proj_{\vec{OD}} \vec{AC} = \dfrac{4+12+2}{3}$
$= 19/3$ $\quad = \dfrac{18}{3}$

$\boxed{-\frac{1}{3} + \frac{19}{3} = 18/3}$:)

c) They are all in the same direction as $\vec{OD}$, and $\vec{AB} + \vec{BC}$ produce $\vec{AC}$.

15a) $\vec{OP} = (x, y, z)$

$LS = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$ $\qquad RS = 1$

$= \left(\frac{x}{|\vec{OP}|}\right)^2 + \left(\frac{y}{|\vec{OP}|}\right)^2 + \left(\frac{z}{|\vec{OP}|}\right)^2$

$= \dfrac{x^2}{x^2+y^2+z^2} + \dfrac{y^2}{x^2+y^2+z^2} + \dfrac{z^2}{x^2+y^2+z^2}$

$= \dfrac{x^2+y^2+z^2}{x^2+y^2+z^2}$ $\qquad \therefore LS = RS$

$= 1$

b) $\cos 30° = \dfrac{y}{|\vec{OP}|}$ $\qquad \cos 60° = \dfrac{z}{|\vec{OP}|}$ $\qquad \cos 90° = \dfrac{x}{|\vec{OP}|}$

$\dfrac{\sqrt{3}}{2} |\vec{OP}| = y$ $\qquad \dfrac{1}{2} |\vec{OP}| = z$ $\qquad 0 = \dfrac{x}{|\vec{OP}|}$

Assume $|\vec{OP}| = 1$: $\vec{OP} = \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ $\quad x = 0$

c) They must add to 180° if two are perpendicular.

Hilroy

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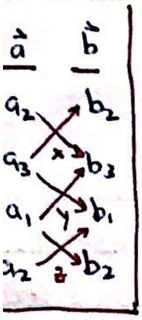
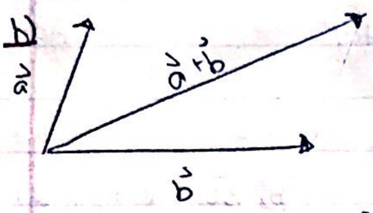


7.6 The Cross Product of Two Vectors

1a) $\vec{a}\times\vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$, so the dot product of a or b with $\vec{a}\times\vec{b}$ will be zero by definition.

b)



$(\vec{a}+\vec{b})\cdot(\vec{a}\times\vec{b})=0$ because $\vec{a}+\vec{b}$ is in the same plane as $\vec{a}$ and $\vec{b}$, and $\vec{a}\times\vec{b}$ is a normal vector to that plane.

c) See explanation for (b)

2. $(\vec{a}\cdot\vec{b})(\vec{a}\times\vec{b})=0$ is impossible because the left side is a vector quantity.

3a) $\vec{a}\cdot(\vec{b}\times\vec{c})$ is possible, it is the dot product of two vectors.

b) $(\vec{a}\cdot\vec{b})\times\vec{c}$ is not possible because $\vec{a}\cdot\vec{b}$ is a scalar quantity.

c) $(\vec{a}\times\vec{b})\cdot(\vec{c}+\vec{d})$ is possible
→ two vectors.

d) $(\vec{a}\cdot\vec{b})(\vec{c}\times\vec{d})$ is possible
→ scalar multiplication

e) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ is possible
→ vector × vector

f) $\vec{a}\times\vec{b}+\vec{c}$ is possible
→ add a vector to a vector

4a) $\vec{a}$: −3, 5, 2, −3; $\vec{b}$: −1, 4, 0, −1

$\vec{a}\times\vec{b}=(-12+5,\ 0-8,\ -2-0)$
$=(-7,-8,-2)$

Check: $(0,-1,4)\cdot(-7,-8,-2)$
$=0+8-8$
$=0 \therefore \vec{a}\times\vec{b}\perp\vec{b}$.

b) $\vec{a}$: −1, 3, 2, −1; $\vec{b}$: −1, 2, 3, −1

$\vec{a}\times\vec{b}=(-2+3,\ 9-4,\ -2+3)$
$=(1,5,1)$

c) $\vec{a}\times\vec{b}=(-7-4,\ 2-35,\ 20+2)$
$=(-11,-33,22)$

d) $\vec{a}\times\vec{b}=(8-27,\ -18-4,\ 3+4)$
$=(-19,-22,7)$

e) $\vec{a}\times\vec{b}=(0+3,\ 3-0,\ 2-3)$
$=(3,3,-1)$

f) $\vec{a}\times\vec{b}=(4-12,\ -6-20,\ 10+1)$
$=(-8,-26,11)$

5. $\vec{a}$: 3, 5, −1, 3; $\vec{b}$: a, 1, 0, a

$(3-5a,\ 0+1,\ -a-0)=(-2,1,-1)$
$(3-5a,\ 1,\ -a)=(-2,1,-1)$
$\therefore a=1$

6a) $\vec{a}\times\vec{b}=(1-5,\ 0-0,\ 0-0)$
$=(-4,0,0)$

b) Vectors of the form $(0,b,c)$ are in the yz plane, so the cross product of two vectors in the xy-plane would have to be in the x direction.

7a) $\vec{a}\times\vec{b}=(4-4,\ 2-2,\ 4-4)$
$=\vec{0}$

b) $\vec{a}\times\vec{b}=(kbc-kbc,\ kac-kac,\ kab-kab)$
$=(0,0,0)$
$=\vec{0}$

8a) $\vec{q}+\vec{r} = (0, 3, 7)$

$\vec{p}\times(\vec{q}+\vec{r}) = (-14-12, 0-7, 3+0)$
$= (-26, -7, 3)$

$\vec{p}\times\vec{q} = (-14-8, 4-7, 2+2)$
$= (-22, -3, 4)$

$\vec{p}\times\vec{r} = (0-4, -4, 1-2)$
$= (-4, -4, -1)$

$(\vec{p}\times\vec{q})+(\vec{p}\times\vec{r}) = (-26, -7, 3)$

$\therefore \vec{p}\times(\vec{q}+\vec{r}) = (\vec{p}\times\vec{q})+(\vec{p}\times\vec{r})$

9a) $\vec{i}\times\vec{j} = (0, 0, 1) = \vec{k}$ $\quad -\vec{j}\times\vec{i} = (0, 0, 0+1) = (0, 0, 1) = \vec{k}$

b) $\vec{j}\times\vec{k} = (1, 0, 0) = \vec{i}$ $\quad -\vec{k}\times\vec{j} = (0+1, 0, 0) = (1, 0, 0) = \vec{i}$

c) $\vec{k}\times\vec{i} = (0, 1, 0) = \vec{j}$ $\quad -\vec{i}\times\vec{k} = (0, 1, 0) = \vec{j}$

10. The cross product of $\vec{a}$ and $\vec{b}$ (or a scalar multiple of $\vec{a}$ or $\vec{b}$) will be $\perp$ to both $\vec{a}$ and $\vec{b}$, so taking the dot product of $k(\vec{a}\times\vec{b})$ and $\vec{a}$ will result in an answer of zero.

11a) $\vec{a}\times\vec{b} = (0, 0, 6)$

$\vec{c}\times\vec{d} = (0, 0, -6)$

b) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$

c) $\vec{a}$ + $\vec{b}$ are parallel to the xy axes, so $\vec{a}\times\vec{b}$ is parallel to the z axis. $\vec{c}$ + $\vec{d}$ are in the xy-plane, so $\vec{c}\times\vec{d}$ is parallel to the z-axis as well. The cross product of parallel vectors is zero.

8b) $\vec{q}+\vec{r} = (3, 2, 1)$

$\vec{p}\times(\vec{q}+\vec{r}) = (1-4, 6-4, 8-3)$
$= (-3, 2, -5)$

$\vec{p}\times\vec{q} = (-1-2, 6+4, 4-3)$
$= (-3, 10, 1)$

$\vec{p}\times\vec{r} = (2-2, 0-8, 4-0)$
$= (0, -8, 4)$

$(\vec{p}\times\vec{q})+(\vec{p}\times\vec{r}) = (-3, 2, 5)$

$\therefore \vec{p}\times(\vec{q}+\vec{r}) = (\vec{p}\times\vec{q})+(\vec{p}\times\vec{r})$

7.7 Applications of the Dot and Cross Product

1. $|\vec{r} \times \vec{f}| = (|\vec{r}|\sin\theta)|\vec{f}|$ — $\theta = 90°$ if you push ⊥ to the door, so $\sin\theta = 1$ (largest value)

$|\vec{r}|$ increases as you get further from the hinges

2a) $\vec{a} \times \vec{b} = (0,0,0)$

b) $\vec{b} = 2\vec{a}$, so they form a line.

4. See 7.6.

5a) $|\vec{a} \times \vec{b}| = |(1,-1,-1)|$
$= \sqrt{3}$ sq. units

b) $|\vec{a} \times \vec{b}| = |(-14,-1,4)|$
$= \sqrt{213}$ sq. units

6. $|\vec{p} \times \vec{q}| = |(3, -1-2a, a-1)|$
$= \sqrt{9+(1+4a+4a^2)+(a^2-2a+1)}$
$= \sqrt{9+5a^2+2a+2}$
$\sqrt{35} = \sqrt{5a^2+2a+11}$
$\therefore 5a^2+2a+11 = 35$
$5a^2+2a-24=0$
$(5a+12)(a-2)=0$
$a = -12/5$ or $a = 2$

3. $W = \vec{F} \cdot \vec{s}$

a) $W = (150\text{ N})(3\text{ m})\cos 0°$
$= 450\text{ N}\cdot\text{m}$ (J)

b) $W = (392\text{N})(40\text{m})\cos 50°$
$= 10\ 078.91$ J

c) $W = (140\text{N})(250\text{m})\cos 20°$
$= 32\ 889.24$ J

d) $W = (100\text{N})(500\text{m})\cos 45°$
$= 25000\sqrt{2}$ J (35 355.34 J)

7a) $\vec{AB} = (3,-1,-2)$ $\vec{AC} = (4,2,-1)$
$\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}|(5,-5,10)|$
$= \frac{1}{2}\sqrt{150}$
$= \frac{5\sqrt{6}}{2}$ sq. units

b) $\vec{BC} = (1,3,1)$ $\vec{CA} = (-4,2,1)$
$\frac{1}{2}|\vec{BC} \times \vec{CA}| = \frac{1}{2}|(5,-5,10)|$
$= \frac{5\sqrt{6}}{2}$ sq. units

c) The area is the same regardless of the sides that we find the vecter product of.

8. $|\vec{r} \times \vec{F}| = |\vec{r}||\vec{f}|\sin\theta$
$= (0.14)(10)\sin 45°$
$= 1.4\left(\frac{\sqrt{2}}{2}\right)$
$= 0.7\sqrt{2}$
$= 0.99$ J